

≠ ECE 307 HOMEWORK 2 ≠

① We toss a fair coin two times.

$S = \{hh, ht, th, tt\}$ → sample space of the experiment

S_i → denotes a simple outcome

We define the random variable \tilde{x} on simple outcomes as follows

$$\tilde{x}(S_i) = \{2 \times \text{number of heads in } S_i - 1 \times \text{number of tails in } S_i\}$$

a) Find all values of the random variable \tilde{x}

b) $\overset{a}{\Leftrightarrow} \{S_i / \tilde{x}(S_i) = 1\}$

$\overset{b}{\Leftrightarrow} \{S_i / \tilde{x}(S_i) = -2\}$

$\overset{c}{\Leftrightarrow} \{S_i / \tilde{x}(S_i) \leq 1\}$

c) Find the prob. mass function

$$P(x) = \text{Prob}(\tilde{x} = x)$$

Draw the graph of $P(x)$

d) $F(x) = \text{Prob}(\tilde{x} \leq x)$

Find cumulative distribution function $F(x)$ and draw the graph of $F(x)$

Solution:

a) $\tilde{x}(hh) = 2 \cdot 2 - 1 \cdot 0 = 4 \quad (S_1)$

$\tilde{x}(ht) = 2 \cdot 1 - 1 \cdot 1 = 1 \quad (S_2)$

$\tilde{x}(th) = 2 \cdot 1 - 1 \cdot 1 = 1 \quad (S_3)$

$\tilde{x}(tt) = 2 \cdot 0 - 1 \cdot 2 = -2 \quad (S_4)$

b) $a = \{S_i / \tilde{x}(S_i) = 1\} \rightarrow a = \{S_2, S_3\} = \{ht, th\}$

$b = \{S_i / \tilde{x}(S_i) = -2\} \rightarrow b = \{S_4\} = \{tt\}$

$c = \{S_i / \tilde{x}(S_i) \leq 1\} \rightarrow c = \{S_2, S_3, S_4\} = \{ht, th, tt\}$

c) x can take values $-2, 1, 4$

$$P(-2) = \text{Prob}(\tilde{x} = -2)$$

$$= \text{Prob}\{tt\}$$

$$= 1/4$$

$$P(1) = \text{Prob}(\tilde{x} = 1)$$

$$= \text{Prob}(ht, th)$$

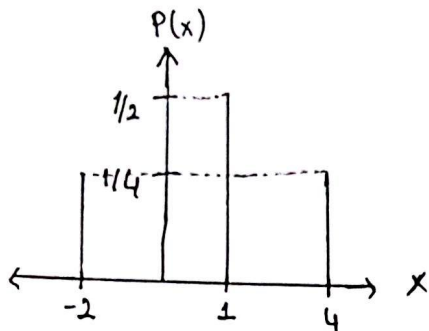
$$= 1/4 + 1/4 = 1/2$$

$$P(4) = \text{Prob}(\tilde{x} = 4)$$

$$= \text{Prob}(hh)$$

$$= 1/4$$

The graph of probb mass funct.



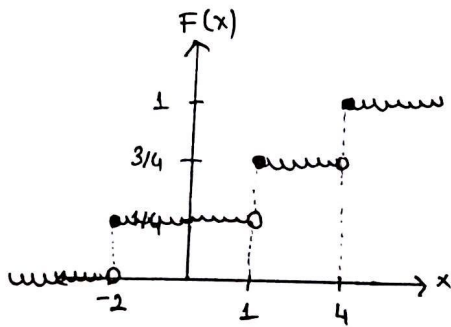
$$P(-2) + P(1) + P(4) = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = \frac{4}{4} = 1$$

d) $F(x) = \text{Prob}(\tilde{x} \leq x)$

$$-\infty < x < -2 \quad \rightarrow \quad F(x) = \text{Prob}(\tilde{x} \leq x) = \text{Prob}(\emptyset) = 0$$

$$-2 \leq x < 1 \quad \rightarrow \quad F(x) = \text{Prob}(\tilde{x} \leq x) = \text{Prob}(tt) = \frac{1}{4}$$

$$4 \leq x < \infty \quad \rightarrow \quad F(x) = \text{Prob}(\tilde{x} \leq x) = \text{Prob}\{hh, tt, h+th\} = 1$$



The graph of cumulative distribution function

② Experiment: Toss of a fair and biased coin

For a fair coin $P(h) = 1/2$ $P(t) = 1/2$

For a biased coin $P(h_b) = 2/3$ $P(t_b) = 1/3$

Sample Space $S = \{hh_b, ht_b, th_b, tt_b\}$

$\tilde{X}(S_i) = \{2 \times \text{number of heads} + 3 \times \text{number of biased tails}\}$

a) Find all values of the random variable \tilde{X}

b) Find the prob. mass function / $P(x) = \text{Prob}(\tilde{X} = x)$

c) Find cumulative distribution function $F(x)$ and draw the graph of $F(x)$

Solution

$$a) \tilde{X}(hh_b) = 2 \times 2 + 3 \cdot 0 = 4$$

$$\tilde{X}(ht_b) = 2 \times 1 + 3 \cdot 1 = 5$$

$$\tilde{X}(th_b) = 2 \times 1 + 3 \cdot 0 = 2$$

$$\tilde{X}(tt_b) = 2 \times 0 + 3 \cdot 1 = 3$$

b) x can take values 2, 3, 4, 5

$$P(2) = \text{Prob}(\tilde{X} = 2)$$

$$= \text{Prob}(th_b)$$

$$= \frac{1}{2} \cdot \frac{2}{3} = 1/3$$

$$P(3) = \text{Prob}(\tilde{X} = 3)$$

$$= \text{Prob}(tt_b)$$

$$= \frac{1}{2} \cdot \frac{1}{3} = 1/6$$

$$P(4) = \text{Prob}(\tilde{X} = 4)$$

$$= \text{Prob}(hh_b)$$

$$= \frac{1}{2} \cdot \frac{2}{3} = 1/3$$

$$P(5) = \text{Prob}(\tilde{X} = 5)$$

$$= \text{Prob}(ht_b)$$

$$= \frac{1}{2} \cdot \frac{1}{3} = 1/6$$

$$c) -\infty < x < 2 \quad \rightarrow \quad F(x) = \text{Prob}(\tilde{X} \leq x) \\ = \text{Prob}(\emptyset) = 0$$

$$2 \leq x < 3 \quad \rightarrow \quad F(x) = \text{Prob}(\tilde{X} \leq x) \\ = \text{Prob}(th_b) \\ = 1/3$$

$$3 \leq x < 4 \quad \rightarrow \quad F(x) = \text{Prob}(\tilde{X} \leq x) \\ = \text{Prob}(th_b, tt_b) \\ = 1/3 + 1/6 = 1/2$$

$$\begin{aligned}
 4 \leq x < 5 &\longrightarrow F(x) = \text{Prob}(\tilde{x} \leq x) \\
 &= \text{Prob}(thb, ttb, htb) \\
 &= 1/3 + 1/6 + 1/3 \\
 &= 5/6
 \end{aligned}$$

$$\begin{aligned}
 5 \leq x < \infty &\longrightarrow F(x) = \text{Prob}(\tilde{x} \leq x) \\
 &= \text{Prob}(thb, ttb, htb, htb) \\
 &= 1/3 + 1/6 + 1/3 + 1/6 \\
 &= 1
 \end{aligned}$$

The graph of cumulative funct.

