(1)

Example 1

$\tilde{X}$ – RV

$$f_{\tilde{X}}(x) = \begin{cases} 2x/3 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{y} = \tilde{X}^2$ $\quad E(\tilde{y}) = ?$ $\quad Var(\tilde{y}) = ?$

Sln:

$$E(\tilde{y}) = \int g(x) f_{\tilde{X}}(x)\,dx$$

$\tilde{y} = g(\tilde{X})$

$$E(\tilde{y}) = \int_1^2 x^2 \left(\frac{2x}{3}\right) dx$$

$$= \frac{2}{3} \int_1^2 x^3\,dx$$

$$= \frac{2}{3} \left.\frac{x^4}{4}\right|_1^2$$

$$= \frac{2}{3}\left(\frac{1}{4}\right)(2^4 - 1^4)$$

$$= \frac{2}{3} \cdot \frac{1}{4} \cdot 15$$

$$= 5/2$$

$$E(\tilde{y}^2) = \int y^2 f_{\tilde{X}}(x)\,dx$$

$$= \int g^2(x) f(x)\,dx$$

$$= \int_1^2 (x^2)^2 \frac{2x}{3}\,dx$$

$$= \frac{2}{3}\int_1^2 x^5\,dx$$

$$= \frac{2}{3} \left.\frac{x^6}{6}\right|_1^2$$

$$= \frac{1}{9}(64 - 1)$$

$$= \frac{63}{9} = 7$$

$$Var(\tilde{y}) = E(\tilde{y}^2) - (E(\tilde{y}))^2$$

$$= 7 - \left(\frac{5}{2}\right)^2$$

②

Ex2 C.D.F of $\tilde{X}$ is

$$F_{\tilde{X}}(x) = \begin{cases} 1 - \frac{a^3}{x^3} & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}$$

$\tilde{X} \to$ cont. RV.

$f_{\tilde{X}}(x) = ?$ $E(\tilde{X}) = ?$ $Var(\tilde{X}) = ?$

Sln2 $f_{\tilde{X}}(x) = \frac{d}{dx} F_{\tilde{X}}(x)$ $E(\tilde{X}) = \int x f(x) dx$

$Var(\tilde{X}) = E(\tilde{X}^2) - (E(\tilde{X}))^2$

$E(\tilde{X}^2) = \int x^2 f(x) dx$

Ex3 $\tilde{X} \sim N(1, 4)$ $\tilde{Y} = 2\tilde{X} + 3$

variance

mean

normal

a) $f_{\tilde{Y}}(y) = ?$ b) Find $P(\tilde{Y} \geq 0)$

Sln3 if $\tilde{X} \sim N(m, \sigma^2)$ then

$\tilde{Y} = a\tilde{X} + b$ is also normal

with mean $m+b$ and variance $a^2\sigma^2$

i.e, $\tilde{Y} \sim N(b+m, a^2\sigma^2)$

③ Thus

$$f_{\tilde{x}}(x) = \frac{1}{\sqrt{2\pi}\,2} e^{-(x-1)^2/2.4}$$

a)

then $$f_{\tilde{y}}(y) = \frac{1}{\sqrt{2\pi}\,4} e^{-(x-4)^2/2.16}$$

b) $p(\tilde{y} \geq 0)$? $\quad \tilde{y} \sim N(4, 16)$

$$p(0 \leq \tilde{y} \leq \infty) = ?$$

$$p(a \leq \tilde{y} \leq b) = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)$$

$$p(0 \leq \tilde{y} \leq \infty) = \Phi(\infty) - \Phi\left(\frac{0-4}{4}\right)$$

$$= \Phi(\infty) - \Phi(-1)$$

$$= 1 - \underbrace{\Phi(-1)}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

$\Phi(\infty) = 1$

$\Phi(-\infty) = 0$

its value is found from tables in books or computed numerically using matlab

(4)

<u>Ex 4</u>

$\tilde{X} \to$ cont. R.V. with PDF

$$f_{\tilde{X}}(x) = \begin{cases} 2x/3 & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and let A be the event $\{\tilde{X} \geq 1.5\}$.

Calculate $E(\tilde{X})$, $P(A)$, $E(\tilde{X}|A)$

<u>Sln 4</u>

$$E(\tilde{X}) = \int x\, f(x)\, dx \to E(\tilde{X}) = \int_1^2 (x)\, \frac{2x}{3}\, dx$$

$$= \frac{2}{3} \frac{x^3}{3} \Big|_1^2$$

$$= \frac{2}{3} \frac{7}{3}$$

$$= 14/9$$

$$P(A) = P(\tilde{X} \geq 1.5) = P(1.5 \leq \tilde{X} \leq \infty)$$

$$= \int_{1.5}^{2} f_{\tilde{X}}(x)\, dx = \int_{1.5}^{2} \frac{2x}{3}\, dx = \frac{2}{3} \frac{x^2}{2} \Big|_{1.5}^{2}$$

$$= \frac{1}{3}(2-1.5)(3.5)$$

$$= \frac{3.5}{6}$$

<u>Note:</u>

$$P(a \leq \tilde{X} \leq b) = \int_a^b \underbrace{f_{\tilde{X}}(x)}_{\text{p.d.f}}\, dx$$

cont. R.V.

$$= \underbrace{F_{\tilde{X}}}_{\text{CDF}}(b) - F_{\tilde{X}}(a)$$

⑤

$$E(\tilde{x}|A) = E(\tilde{x}|\ \tilde{x} \geq 1.5)$$
$$= \int_{-\infty}^{\infty} x\, f_{\tilde{x}|A}(x)\, dx$$

$$f_{\tilde{x}|A}(x) = \begin{cases} \dfrac{f_{\tilde{x}}(x)}{P(A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{2x/3}{3.5/6} \Rightarrow \dfrac{2x}{3} \cdot \dfrac{6}{3.5} \Rightarrow \dfrac{4x}{3.5} & \text{if } 1.5 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $E(\tilde{x}|A) = \displaystyle\int_{1.5}^{2} x \frac{4x}{3.5}\, dx$

$$= \frac{4}{3.5} \left.\frac{x^3}{3}\right|_{1.5}^{2}$$

$$= \frac{4}{3.5} \cdot \frac{1}{3}(8 - 1.5^3)$$

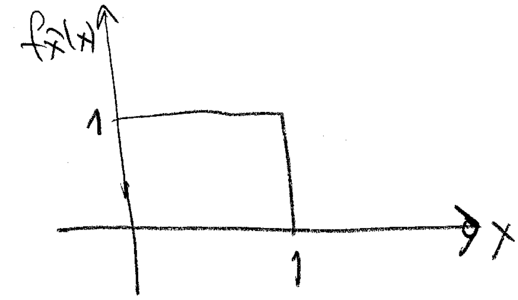
Ex5 $\tilde{x}$ & $\tilde{y}$ be independent RVs, with each one uniformly distributed in the interval $[0,1]$.
Find the prob. of each of the following events

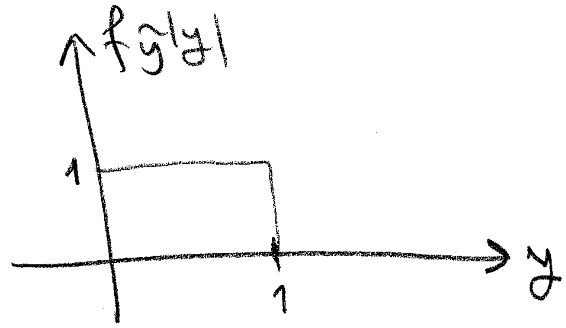
a) $\tilde{x} > 6/10$ b) $\tilde{y} < \tilde{x}$ c) $\tilde{x} + \tilde{y} \leq 3/10$

d) $\max(\tilde{x}, \tilde{y}) \geq 1/3$ e) $\tilde{x}\tilde{y} \leq 1/4$

⑥

$\underline{\underline{5/n5}}$ $\tilde{x}$ & $\tilde{y}$ are uniformly distributed continuous (on $[0,1]$)
RVs (independent)





Since $\tilde{x}$ & $\tilde{y}$ are independent

$$f_{\tilde{x},\tilde{y}}(x,y) = f_{\tilde{x}}(x)\, f_{\tilde{y}}(y)$$

→ joint p.d.f. of $\tilde{x}$ & $\tilde{y}$

a) $P\left(\tilde{x} > \frac{6}{10}\right) = P\left(\frac{6}{10} < \tilde{x} < \infty\right) = \int_{6/10}^{\infty} f_{\tilde{x}}(x)\,dx$

$= \int_{6/10}^{1} 1\,dx$

$= 1 - \frac{6}{10} = 4/10$

b) $P(\tilde{y} < \tilde{x}) = P(\tilde{y} - \tilde{x} < 0) = \iint_{y-x<0} f(x,y)\,dy\,dx$

x → Values that RV $\tilde{x}$ can take

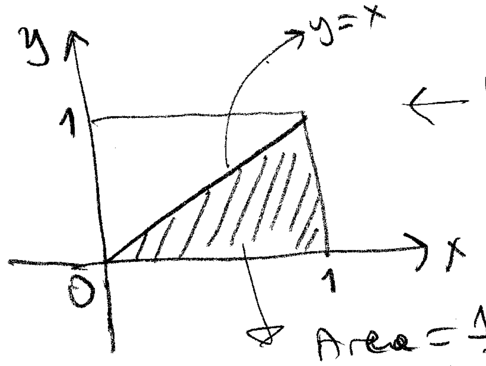
y → Values that RV $\tilde{y}$ can take.

$= \int_0^1 \int_0^x \underbrace{f(x)}_{1} \underbrace{f(y)}_{1}\,dy\,dx$

$= \int_0^1 \int_0^x dy\,dx$

$= \int_0^1 x\,dx = \int_0^1 \frac{x^2}{2} = \frac{1}{2}$

7



← (x, y) plane

$y < x$

Since $f(x,y) = f(x)f(y) = 1$

it is sufficient to consider the area below $y=x$

c) $$P(\hat{x}+\hat{y} \le \tfrac{3}{10}) = \iint_{x+y \le \frac{3}{10}} f_{\hat{x},\hat{y}}(x,y)\,dy\,dx$$

$\hat{x}$ & $\hat{y}$ are independent
$f(x,y) = f(x)f(y)$

$$= \int_0^1 \int_0^{3/10 - x} f(x)f(y)\,dy\,dx$$

$$= \int_0^1 \int_0^{3/10 - x} 1 \cdot 1 \,dy\,dx$$

$$= \int_0^1 \left(\frac{3}{10} - x\right) dx$$

$$= \frac{3}{10}x - \frac{x^2}{2} \Big|_0^1$$

$$= 3/10 - 1/2$$

$$= -2/10$$

$$= -1/5$$

d) $\max\{X, Y\} \geq \frac{1}{3}$

$$P\left(\max\{X,Y\} \geq \tfrac{1}{3}\right) = 1 - P\left(\max(X,Y) \leq \tfrac{1}{3}\right)$$

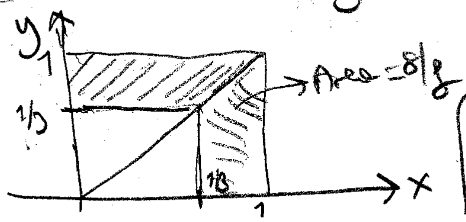
$$P\left(\max(X,Y) \leq \tfrac{1}{3}\right) = P\left(X \leq \tfrac{1}{3}, Y \leq \tfrac{1}{3}\right)$$

$$= P\left(X \leq \tfrac{1}{3}\right) \cdot P\left(Y \leq \tfrac{1}{3}\right) \rightarrow X \,\&\, Y \text{ are independent}$$

$$= F_X\left(\tfrac{1}{3}\right) F_Y\left(\tfrac{1}{3}\right)$$

$$= \tfrac{1}{3} \cdot \tfrac{1}{3}$$

$$P\left(\max(X,Y) \geq \tfrac{1}{3}\right) = 1 - \tfrac{1}{9} = 8/9$$



OR

$$P\left(\max(X,Y) \geq \tfrac{1}{3}\right) = \iint_{\max(x,y) \geq \frac{1}{3}} f(x,y)\,dx\,dy$$

$$= \int_{1/3}^{1}\int_{0}^{x^*} 1 \cdot 1\,dy\,dx + \int_{1/3}^{1}\int_{0}^{y} 1 \cdot 1\,dx\,dy$$

$$= 8/9$$

Ex 6: Two cont. RVs X & Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } x>0 \,\&\, y>0 \,\&\, x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let A be the event $\{Y \leq 0.5\}$ and let B the event $\{Y \geq X\}$

a) Calculate $P(B|A)$

b) $f_{X|B}(x) = ?$

c) $E(XY) = ?$

d) Calculate p.d.f. of Y/X

9

Sln 5

$P(B|A) = P(\tilde{y} \geq \tilde{x}^2 \mid \tilde{y} \leq 0.5)$

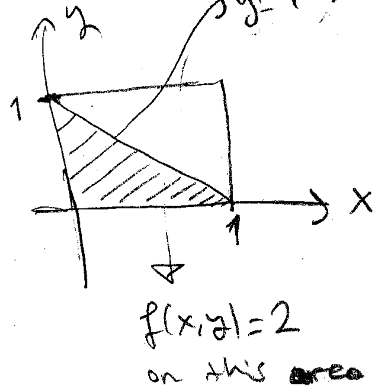
$P(B|A) = \dfrac{P(B \overset{\cap}{,} A)}{P(A)}$

$= \dfrac{P(\tilde{y} \geq \tilde{x} \overset{\cap}{,} \tilde{y} \leq 0.5)}{P(\tilde{y} \leq 0.5)}$

$(\tilde{y} \geq \tilde{x}, \tilde{y} \leq 0.5) = (\tilde{x} \leq \tilde{y} \leq 0.5)$

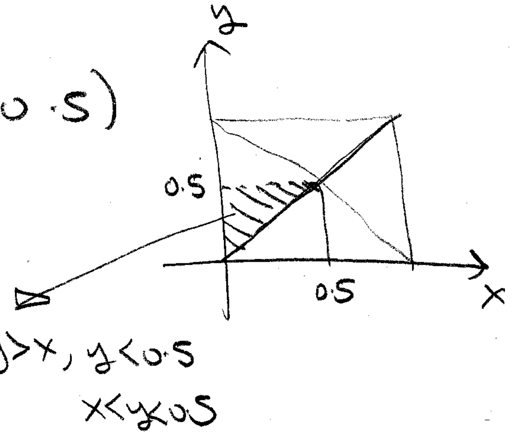
$= \dfrac{P(\tilde{x} \leq \tilde{y} \leq 0.5)}{P(\tilde{y} \leq 0.5)}$

$f(x,y) = \begin{cases} 2 & \text{if } x>0,\ y>0 \ \& \ \underbrace{x+y \leq 1}_{y \leq 1-x} \\ 0 & \text{otherwise} \end{cases}$



$P(B|A) = \dfrac{P(\tilde{x} \leq \tilde{y} \leq 0.5)}{P(\tilde{y} \leq 0.5)}$

$(\tilde{x} \leq \tilde{y} \leq 0.5) = (\tilde{y} \geq \tilde{x}, \tilde{y} \leq 0.5)$



$P(\tilde{x} \leq \tilde{y} \leq 0.5) = \iint\limits_{x<y<0.5} f(x,y)\,dx\,dy$

$= (\text{Shaded Area}) \cdot 2 \Rightarrow f(x,y) = 2$

$= \dfrac{(0.5)(0.5)}{2} \cdot 2$

$= 1/4$

$P(\tilde{y} \leq 0.5) = \int_0^1 \int_0^{0.5} \overbrace{f(x,y)}^{2}\,dy\,dx$

$= \dfrac{1}{2}$

Hence;

$$p(B|A) = \frac{1/4}{1/2} = \frac{1}{2}$$

b) $$f_{\tilde{x}|B}(x) = \begin{cases} \frac{f(x)}{p(B)} & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

$$P(B) = P(\tilde{y} > \tilde{x}) = \int_0^1 \int_x^1 \underbrace{f(x,y)}_{2}\, dy\, dx$$

$$= P(\tilde{y} - \tilde{x} > 0) = \int_0^1 \int_x^1 2\, dy\, dx$$

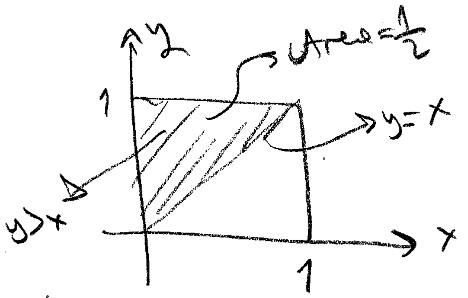
$$= \iint_{y>x} f(x,y)\, dy\, dx = \int_0^1 (2)(1-x)\, dx$$

$$= 2 \int_0^1 (1-x)\, dx$$

$$= 2 \left(x - \frac{x^2}{2} \right) \Big|_0^1$$

$$= (2)\left(1 - \frac{1}{2}\right)$$

$$= 1 //$$



Area $(y>x)$ $\times$ $(f(x,y) = 2)$

$$= \frac{1}{2} \cdot 2 = 1 //$$

$\longrightarrow$ $p = 1 //$ Same

c) $$E(\tilde{x}\tilde{y}) = \iint xy\, f(x,y)\, dx\, dy$$

<u>Note:</u> $$E(g(\tilde{x}, \tilde{y})) = \iint g(x,y)\, f(x,y)\, dx\, dy$$

Hence $$E(\tilde{x}, \tilde{y}) = \int_0^1 \int_0^1 xy\, 2\, dx\, dy$$

11

d) P.d.f. of $\tilde{y}/\tilde{x}$?

Let $\tilde{z} = \tilde{y}/\tilde{x}$

C.DF of $\tilde{z}$ $\quad F_{\tilde{z}}(z) = P(\tilde{z} \le z)$

$$F_{\tilde{z}}(z) = P(\tilde{z} \le z)$$

$$= P\left(\frac{\tilde{y}}{\tilde{x}} \le z\right)$$

$$= P(\tilde{y} \le \tilde{x}z, \tilde{x} > 0) + \cancel{P(\tilde{y} \ge \tilde{x}z, \tilde{x} < 0)}$$

$$= \iint_{\substack{y<xz \\ x>0}} \underbrace{f(x,y)}_{=2} \, dy\, dx + \cancel{\iint_{\substack{y>xz \\ x<0}} \underbrace{f(x,y)}_{=2} \, dy\, dx}$$

$$= \int_0^1 \int_0^{xz} 2 \, dy\, dx + \cancel{\int_0^1 \int_{xz}^1 2 \, dy\, dx}$$

Since x is always >0 in question.

$$= \int_0^1 \int_0^{xz} 2 \, dy\, dx$$

$$= \int_0^1 (2)(xz)\, dx$$

$$= (2z) \left. \frac{x^2}{2} \right|_0^1$$

$$= z$$

Hence:

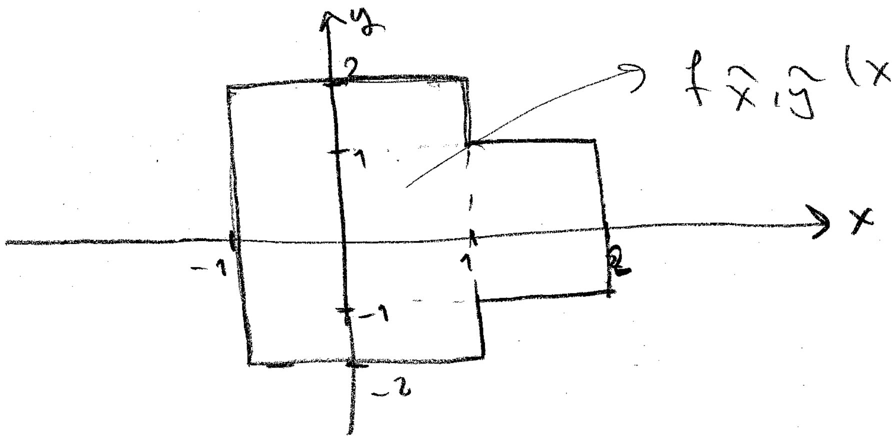
$F_{\tilde{z}}(z) = z \qquad z > 0$

Since $y>0$, $x>0$, $z = y/x > 0$

$$f_{\tilde{z}}(z) = \frac{dF(z)}{dz} = 1$$

Thus: $\tilde{z}$ is also uniform RV with p.d.f  $\underline{0 \le z \le 1}$ //

Ex 7: The RVs $\tilde{X}$ & $\tilde{Y}$ have the joint PDF shown below



a) Find the conditional PDFs $f_{\tilde{Y}|\tilde{X}}(y|x)$

and $f_{\tilde{X}|\tilde{Y}}(x|y)$ for various values of x and y

b) Find $E(\tilde{X}|\tilde{Y})$ and $Var(\tilde{X}|\tilde{Y})$.

$E(\tilde{X})$

c) Find $E(\tilde{Y}|\tilde{X})$ and $Var(\tilde{Y}|\tilde{X})$.

$E(\tilde{Y})$

Sln 7: $f(x,y)$ can also be written mathematically as

$$f(x,y) = \begin{cases} 0.1 & \text{if } -1 \le x \le 1 \text{ and } -2 \le y \le 2 \\ 0.1 & \text{if } 1 \le x \le 2 \text{ and } -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a). $f_{\tilde{X}|\tilde{Y}}(x|y) = \dfrac{f_{\tilde{X},\tilde{Y}}(x,y)}{f_{\tilde{Y}}(y)}$ $\qquad f_{\tilde{Y}}(y) = \int f(x,y)\,dx$

⑬

$$f_Y(y) = \int f(x,y)\,dx \rightarrow f_Y(y) = \int_{-1}^{1} 0.1\,dx \qquad -2 \le y \le 2$$

$$= 0.2$$

$$f_Y(y) = \int_{1}^{2} 0.1\,dx \qquad -1 \le y \le 1$$

$$= 0.1$$

Thus

$$f_Y(y) = \begin{cases} 0.2 & -2 \le y \le 2 \\ 0.1 & -1 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Check $\int f(y)\,dy \overset{?}{=} 1 \rightarrow \int_{-2}^{2} 0.2\,dy + \int_{-1}^{1} 0.1\,dy = 0.8 + 0.2$
$= 1$ ✓

Thus $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$$= \begin{cases} \dfrac{f(x,y)}{0.2} & -2 \le y \le 2 \\ \dfrac{f(x,y)}{0.1} & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{0.1}{0.2} & -1 \le x \le 1 \quad -2 \le y \le 2 \\ \dfrac{0.1}{0.1} & +1 \le x < 2 \quad -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(14) Using a similar approach $f(y|x) = \frac{f(x,y)}{f(x)}$

$$f(x) = \int f(x,y)\,dy$$

b) $E(\bar{X}|\bar{Y}) = ?$

We know that $E(\bar{X}|\bar{Y}=y) = \int x\, f_{\bar{X}|\bar{Y}}(x|y)\,dx$

Hence first, lets compute $E(\bar{X}|\bar{Y}=y)$

$E(\bar{X}|\bar{Y}=y) = \int x\, f(x|y)\,dx$ → use previous found $f(x|y)$

$$= \begin{cases} \int_{-1}^{1} x \cdot \frac{1}{2}\,dx & -2 \le y \le 2 \\ \int_{1}^{2} x \cdot \frac{1}{1}\,dx & -1 \le y \le 1 \\ 0 & o/w \end{cases}$$

$$= \begin{cases} \frac{1}{2}\frac{x^2}{2}\Big|_{-1}^{1} & -2 \le y \le 2 \\ \frac{x^2}{2}\Big|_{1}^{2} & -1 \le y \le 1 \\ 0 & o/w \end{cases}$$

$$= \begin{cases} 0 & -2 \le y \le 2 \\ 3/2 & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus $E(\bar{X}|\bar{Y}) = \begin{cases} 3/2 & -1 \le \bar{Y} \le 1 \\ 0 & \text{otherwise} \end{cases}$

15

$$E(\tilde{x}|\tilde{y}) = \begin{cases} 3/2 & -1 \leq \tilde{y} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(\tilde{x}) = E\left(E(\tilde{x}|\tilde{y})\right)$$

$$= \int E(\tilde{x}|\tilde{y}=y)\, f_{\tilde{y}}(y)\, dy$$

$$f_{\tilde{y}}(y) = \int f(x,y)\, dx$$

$$f_{\tilde{y}}(y) = \begin{cases} 0.2 & -2 \leq y \leq 2 \\ 0.1 & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(\tilde{x}) = \int_{-2}^{2} (0).\,0.2\, dy + \int_{-1}^{1} (0.1).\,\frac{3}{2}\, dy$$

$$= 0.3 \,//$$

$E(\tilde{x})$ can also be computed of

$$E(\tilde{x}) = \int x\, f_{\tilde{x}}(x)\, dx \leftarrow f_{\tilde{x}}(x) = \int f(x,y)\, dy$$

Compute it in this way and compare to the previous found result

$$Var(\tilde{x}|\tilde{y}) = E\left(\left(\tilde{x} - E(\tilde{x}|\tilde{y}=y)\right)^2 \middle| \tilde{y}=y\right)$$

(16) $\mathrm{Var}(\tilde{X}|\tilde{Y}) = E\left(\left((\tilde{X} - E(\tilde{X}|\tilde{Y}=y)\right)^2 \middle| \tilde{Y}=y\right)$

$$E(\tilde{X}|\tilde{Y}=y) = \begin{cases} 3/2 & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Var}(\tilde{X}|\tilde{Y}) = E\left((\tilde{X} - \tfrac{3}{2})^2 \middle| \tilde{Y}=y \right) \quad -1 \le y \le 1$$

$$= \int (x - \tfrac{3}{2})^2 f_{\tilde{X}|\tilde{Y}}(x|y)\,dx$$

$$f(x|y) = \begin{cases} 1/2 & -1 \le x \le 1 \quad -2 \le y \le 2 \\ 1 & 1 \le x < 2 \quad -1 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\mathrm{Var}(\tilde{X}|\tilde{Y}) = \int_1^2 (x - \tfrac{3}{2})^2 \cdot 1\,dx$$

$$= \int_1^2 (x^2 - 3x + \tfrac{9}{4})\,dx$$

17

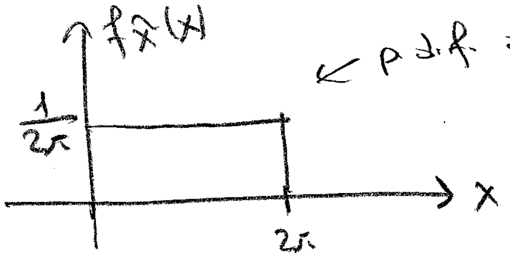
Ex 8:

$\tilde{x} \to$ R.V. uniformly distributed on $[0, 2\pi]$

$\tilde{n} \to$ Normal R.V. $N(0, \sigma^2)$

$\tilde{y} = \tilde{x} + \tilde{n}$ Find $f_{\tilde{x}|\tilde{y}}(x|y)$

Sl/n:



$\tilde{x}$ is cont for $0 \le x \le 2\pi$

$\tilde{n} \sim N(0, \sigma^2)$

$$\tilde{y} = \frac{1}{2\pi} + \tilde{n} \qquad 0 \le x \le 2\pi$$

$$\tilde{y} \sim N\left(\frac{1}{2\pi}, \sigma^2\right)$$

Since if $\tilde{a} \sim N(0, \sigma^2)$

$\tilde{b} = \underbrace{k}_{\text{constant}} + \tilde{a}$

$\tilde{b} \sim N(k, \sigma^2)$

$$f_{\tilde{x}|\tilde{y}}(x|y) = \frac{f_{\tilde{x},\tilde{y}}(x,y)}{f_{\tilde{y}}(y)}$$

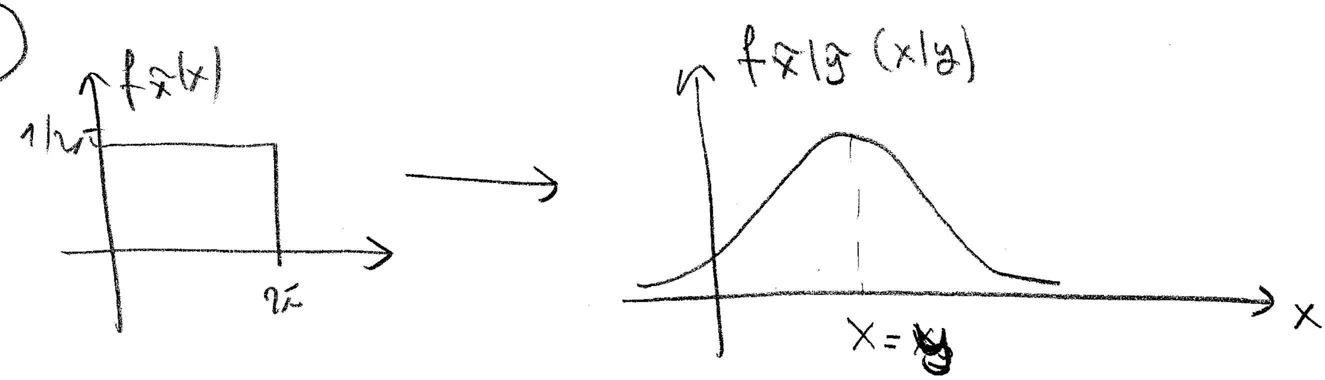
$$= \frac{f(y|x)\, f(x)}{f(y)} = \frac{f(y|x)\, f(x)}{\int f(y|x)\, f(x)\, dx}$$

$$f(y|x) = N(y-x, \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-x)^2/2\sigma^2}$$

$$f(x|y) = \frac{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-x)^2/2\sigma^2} \frac{1}{2\pi}}{\frac{1}{2\pi} \int \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2} dx}$$

$0 \le x \le 2\pi$



Ex9

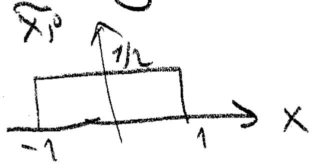
$$f_{\tilde{x}\tilde{y}} = \begin{cases} \frac{2}{\pi} & 0 \le x \le 1,\ 0 \le y \le \frac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(0 \le \tilde{x} \le 0.5,\ 0 \le \tilde{y} \le \pi/4)$

Sln: Solve it $P(a \le \tilde{x} \le b,\ c \le \tilde{y} \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Ex10 $\tilde{y} = \sum_{i=1}^{1000} \tilde{x}_i$ where ~~each~~ x_i are independent RVs each is uniform in the interval $(-1,1)$. Find the probability density function of y

Sln10 According to Central Limit Theorem $\tilde{y}$ approaches Gaussian.



$$E(\tilde{y}) = E\left(\sum_{i=1}^{1000} \tilde{x}_i\right) = \sum_{i=1}^{1000} \underbrace{E(\tilde{x}_i)}_{=0} = 0$$

19

$$Var(\tilde{y}) = \sigma_{\tilde{y}}^2 = \sum_{i=1}^{1000} Var(\tilde{x}_i)$$

$$Var(\tilde{x}_i) = \int_{-1}^{1} (x-0)^2 \, 0.5 \, dx = 1/3$$

$$Var(\tilde{y}) = 1000/3$$

$$f_{\tilde{y}}(y) = \frac{1}{\sqrt{2\pi}\,\sigma_{\tilde{y}}} e^{-(y-m)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sqrt{\frac{1000}{3}}} e^{-y^2/2 \cdot 1000/3}$$

Ex11:

$$f_{\tilde{x},\tilde{y}}(x,y) = \begin{cases} x y^2 e^{-y} & 0 \le y < \infty, \; 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Joint p.d.f of $\tilde{x}$ & $\tilde{y}$

Determine whether $\tilde{x}$ & $\tilde{y}$ are independent or NOT.

Sln11: Solve it $f(x,y) \stackrel{?}{=} f(x)\, f(y)$

check

(20)

Ex12

$\tilde{X} \longrightarrow$ R.V. $\quad F_{\tilde{X}}(x) = 1 - e^{-2x} \quad x \geq 0$

$\tilde{Y} = \tilde{X}^2 \quad F_{\tilde{Y}}(y) = ? \quad f_{\tilde{Y}}(y) = ?$

Sln12

$$F_{\tilde{Y}}(y) = P(\tilde{Y} \leq y)$$
$$= P(\tilde{X}^2 \leq y)$$
$$= P(|\tilde{X}| \leq \sqrt{y})$$
$$= P(-\sqrt{y} \leq \tilde{X} \leq \sqrt{y})$$
$$= F_{\tilde{X}}(\sqrt{y}) - \cancel{F_{\tilde{X}}(-\sqrt{y})}$$

since $x \geq 0$ in $F_{\tilde{X}}(x)$.

$$F_{\tilde{Y}}(y) = (1 - e^{-2\sqrt{y}}) \quad y \geq 0$$

$$f_{\tilde{Y}}(y) = \frac{dF_{\tilde{Y}}(y)}{dy} = \frac{1}{\sqrt{y}} e^{-2\sqrt{y}} \quad y \geq 0$$

Ex13

$$\tilde{Y} = g(\tilde{X}) = \begin{cases} \tilde{X} - 3 & \text{if } \tilde{X} > 3 \\ 0 & -3 < \tilde{X} \leq 3 \\ \tilde{X} + 3 & \tilde{X} \leq -3 \end{cases}$$

Find $F_{\tilde{Y}}(y)$ in terms of $F_{\tilde{X}}(x)$

Sln13

$$F_{\tilde y}(y) = P(\tilde y \le y)$$

if $\tilde x > 3$ $\quad F_{\tilde y}(y) = P(\tilde x - 3 \le y)$

$$= P(\tilde x \le y+3)$$

$$= F_{\tilde x}(y+3)$$

if $-3 < \tilde x \le 3$ $\quad P(\tilde y = 0) = P(-3 < \tilde x \le 3)$

$$= F_{\tilde x}(3) - F_{\tilde x}(-3)$$

if $\tilde x \le -3$ $\quad F_{\tilde y}(y) = P(\tilde x + 3 \le y)$

$$= F_{\tilde x}(y-3)$$

Note: For Cont. R.V.

$$P(\tilde y = a) = F_{\tilde y}(a) - F_{\tilde y}(a^-)$$

For Disc. R.V.

$$P(\tilde y = a_i) = F_{\tilde y}(a_i) - F_{\tilde y}(a_{i-1})$$

Ex14 $\quad \tilde x \to$ uniform R.V. in the interval [0,1]

Find the density of the RV $\tilde y = -\ln \tilde x$

Sln. $\quad$ Formula $\quad f_{\tilde y}(y) = \sum_i \frac{1}{|dy/dx|_{x_i}} f_{\tilde x}(x_i)$

$y = g(x) \to$ solutions x_i

22

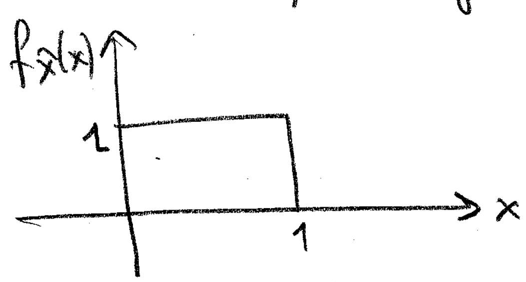
$y = -\ln x \longrightarrow x = e^{-y}$

$g(x) = -\ln x$

$g'(x) = -\frac{1}{x}$ $\qquad f_{\tilde{y}}(y) = \dfrac{f_{\tilde{x}}(e^{-y})}{\frac{1}{e^{-y}}}$

$f_{\tilde{y}}(y) = e^{-y} \underbrace{f_{\tilde{x}}(e^{-y})}_{=1}$ $\qquad \underline{\underline{y > 0}}$

$\tilde{x} \rightarrow$ uniform R.V



Ex 15

$\tilde{x} \sim N(5,4)$

$\tilde{y} = a\tilde{x} + b$ $\qquad E(\tilde{y}) = ?$

$Var(\tilde{y}) = ?$

Sln 15

$E(\tilde{y}) = 5 + b$

$Var(\tilde{y}) = a^2 . 4$

$\underline{\tilde{y} \sim N(b+5, a^2 4)}$

23

Ex15

$\tilde{y} = |\tilde{x}|$

Find $f_{\tilde{y}}(y)$ in terms of $f_{\tilde{x}}(x)$

Sln15 Solve it

Ex16 Show that $F_{\tilde{x}}(\tilde{x}|A) = \dfrac{P(A|\,\tilde{x} \leq x)\, F_{\tilde{x}}(x)}{P(A)}$

Ex17

$\tilde{x} \sim N(0,100)$

a) Find $P(\tilde{x} \leq 0)$

b) Find x at which $F_{\tilde{x}}(x) = \frac{1}{2}$

Ex18:

$\tilde{y} = \sqrt{|\tilde{x}|} \qquad 0 \leq y \leq 1$

$f_{\tilde{y}}(y) = ?$ in terms of $f_{\tilde{x}}(x)$

Sln18

$$F_{\tilde{y}}(y) = P(\tilde{y} \leq y)$$

$$= P(\sqrt{|x|} \leq y) = P(-y^2 \leq x \leq y^2)$$

$$= F_{\tilde{x}}(y^2) - F_{\tilde{x}}(-y^2)$$

(25)

$$f_{\tilde{y}}(y) = \frac{dF_{\tilde{y}}(y)}{dy}$$

$$f_{\tilde{y}}(y) = (2y)\, f_{\tilde{x}}(y^2) + 2y\, f_{\tilde{x}}(-y^2)$$

$$= 4y\, f_{\tilde{x}}(y^2)$$

Ex 18

$$\tilde{y} = |\tilde{x}|^{1/3}$$

$f_{\tilde{y}}(y)$ in terms of $f_{\tilde{x}}(x)$?

Ex 20

$$\tilde{y} = -\ln|\tilde{x}| \qquad y \geq 0$$

$$f_{\tilde{y}}(y) = ?$$

Ex 21

$$f_{\tilde{x}\tilde{y}}(x,y) = \begin{cases} Kxy & 0 \leq x \leq 1 \quad 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Cov(\tilde{x}, \tilde{y}) = ?$$

Sln.

$$Cov(\tilde{x}, \tilde{y}) = \underbrace{E(\tilde{x}\tilde{y})} - E(\tilde{x})\, E(\tilde{y})$$

$\iint f(x,y)\,dx\,dy = 1$ & find k

$$= \iint xy\, f(x,y)\,dx\,dy - \int x f(x)\,dx \int y f(y)\,dy$$

$$f(x) = \int f(x,y)\,dy \quad \boxed{f(y) = \int f(x,y)\,dx}$$

25

Ex 22

$$f_{X,Y}(x,y)=\begin{cases} 1 & 0<|y|<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$E(X|Y)=?$ $\quad E(Y|X)=?$ $\quad E(X)=?$

$E(Y)=?$

$Var(X|Y)=?$

Sln: 22

$$f_X(x)=\int_{-x}^{x} f(x,y)\,dy \qquad 0<x<1$$

$$=2x$$

$$f_Y(y)=\int_{|y|}^{1} 1\,dx=1-|y| \qquad |y|<1$$

$$f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\frac{1}{1-|y|} \qquad 0<|y|<x<1$$

$$E(X|Y=y)=\int x\, f_{X|Y}(x|y)\,dx$$

$$=\int_{|y|}^{1}\frac{x}{(1-|y|)}\,dx=\frac{1+|y|}{2} \qquad |y|<1$$

$$\longrightarrow E(X|Y)=\frac{1+|Y|}{2}$$

$$E(Y|X=x)=\int_{-x}^{x} y\, f_{Y|X}(y|x)\,dy$$

$$=0 \qquad 0<x<1$$