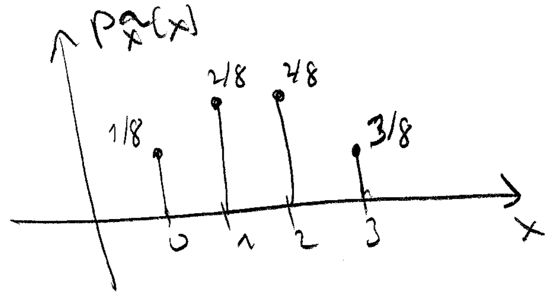
1

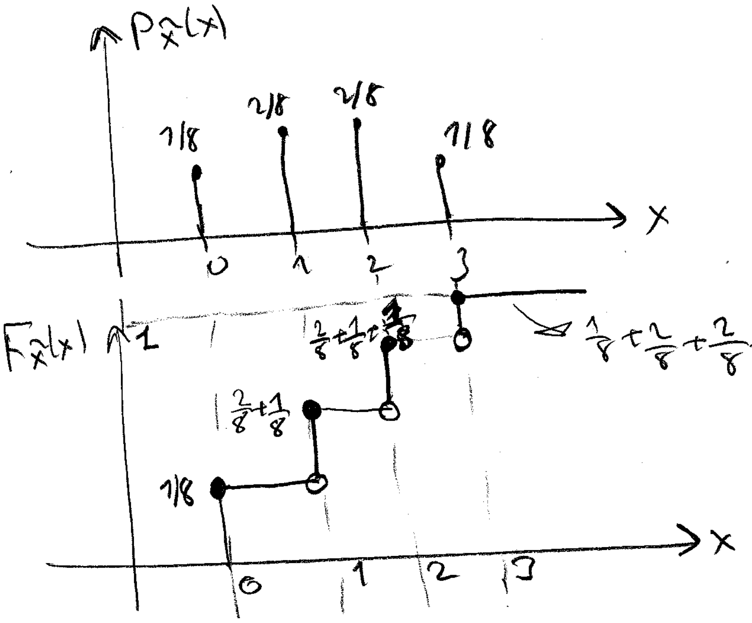
Ex: p.d.f. of disc. R.V. $\hat{x}$ is shown below



Determine C.D.F. of $\hat{x}$, i.e, $F_{\hat{x}}(x) = ?$

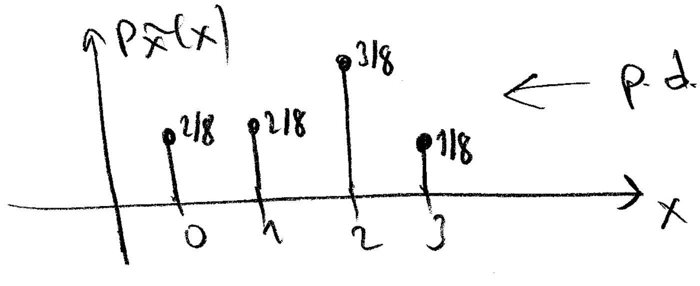
Sln:

$$F(x) = \sum_{k \le x} f(k)$$



$\frac{1}{8}+\frac{2}{8}+\frac{2}{8}+\frac{3}{8}=1$

Ex:



← p.d.f. of $\hat{x}$

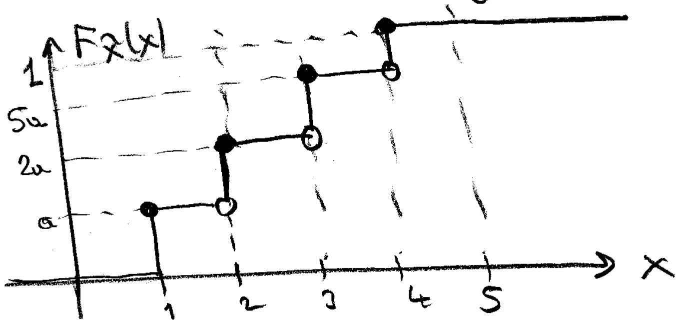
c.d.f of $\hat{x}$?

$F_{\hat{x}}(x)$?

Sln: Solve it

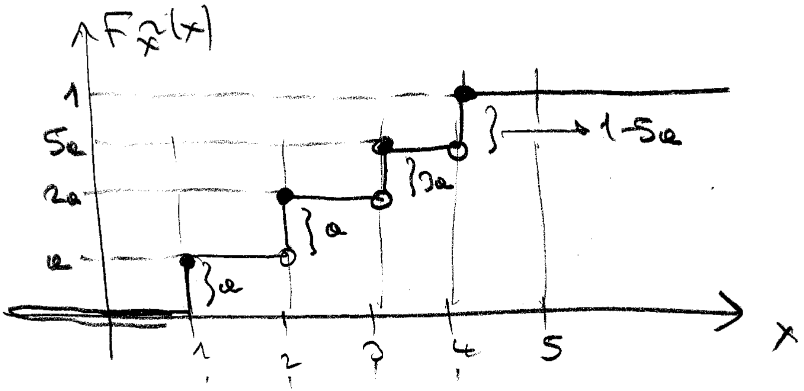
② **Ex:**

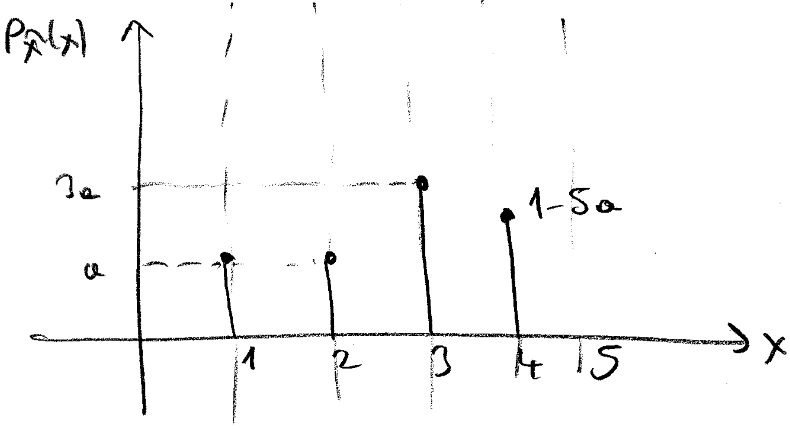
C.D.F of disc R.V. is given below



$P_X(x) = ?$ $\quad a = ?$

Sln:



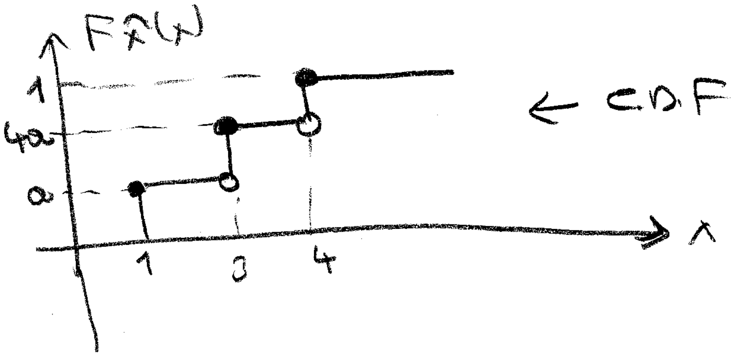


$1-5a > 0 \rightarrow a < 1/5 \qquad a < 0.2$

a can be any real number $0 < a < 0.2$

choose $a = 0.1$ //

(3) **Exe**



$\leftarrow$ CDF of disc R.V $\hat{X}$

p.d.f of $\hat{X}$ $\quad P_{\hat{X}}(x) = ?$ $\quad a = ?$

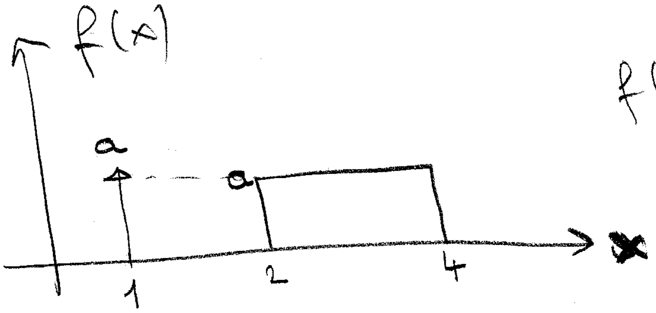
Sln: Solve it.

Cont. R.Vs $\quad f(x) = \dfrac{dF(x)}{dx} \longrightarrow$ c.d.f $\quad F(x) = \int_{-\infty}^{x} f(t)\,dt$

$\downarrow$

pdf

Exe



$f(x) \to$ p.d.f of cont. R.V. $\hat{X}$

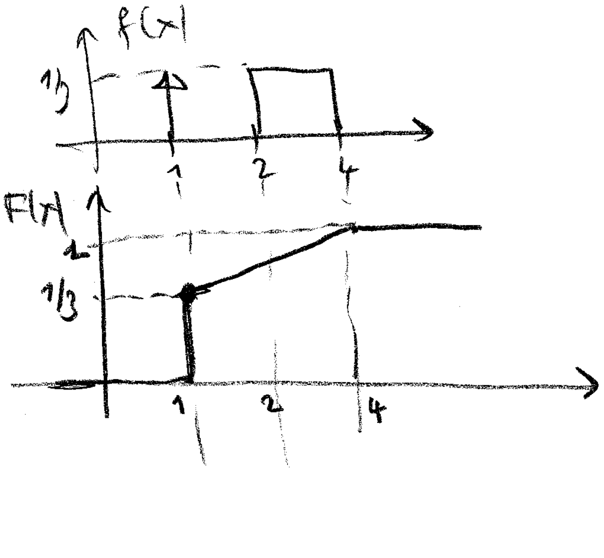
$F_{\hat{X}}(x) \to ?$

c.d.f of $\hat{X}$

$a = ?$

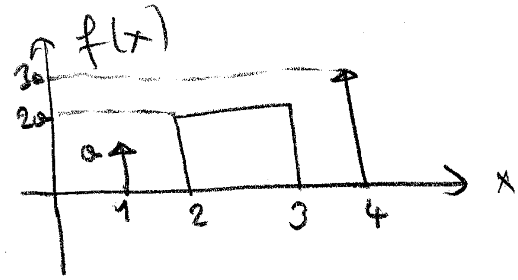
Sln: $\int_{-\infty}^{\infty} f(x)\,dx = 1 \to a + (2)a = 1 \to a = 1/3$

$F(x) = \int_{-\infty}^{x} f(t)\,dt$



④

Exe

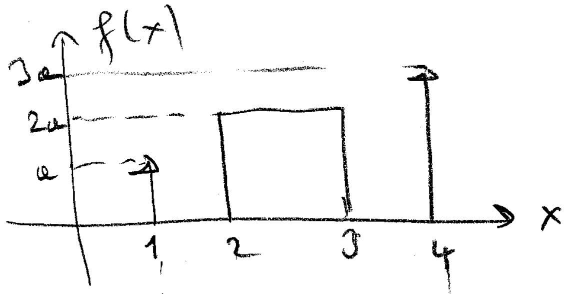


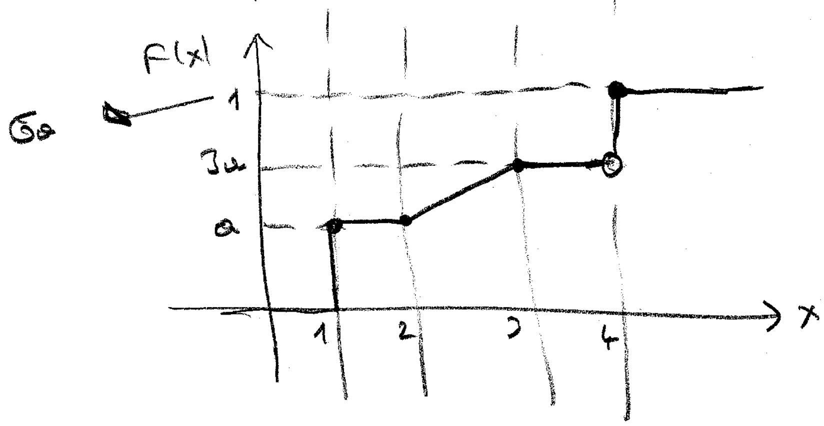
$F(x) = ?$

Sln.

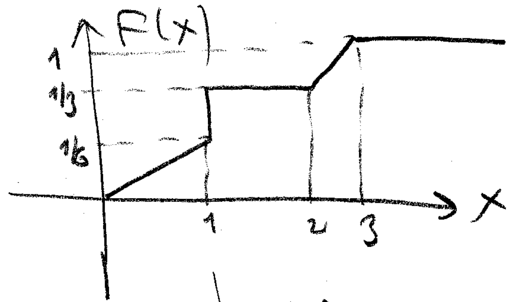
$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow a + (1)(2a) + 3a = 1 \rightarrow 6a = 1$$

$$a = 1/6$$





Exe



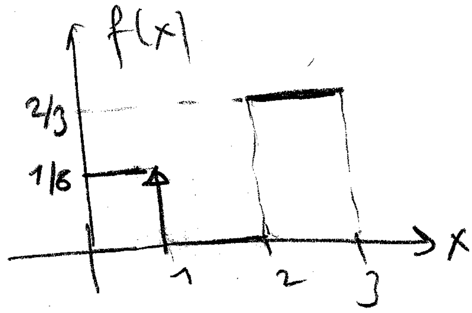
c.d.f

$f(x) = ?$

p.d.f

Sln.

$$f(x) = \frac{d\,F(x)}{d\,x}$$



⑤ **Ex:**

A cont. R.V. $\tilde{X}$ has p.d.f

$$f(x)=\frac{c}{x^2+1} \quad -\infty<x<\infty$$

a) $c=?$ b) $P\left(\frac{1}{3}\leq \tilde{X}^2\leq 1\right)$

Sln:

$$\int_{-\infty}^{\infty} f(x)\,dx=1 \longrightarrow \int_{-\infty}^{\infty} \frac{c}{x^2+1}\,dx=1$$

a)

$$c\int_{-\infty}^{\infty}\frac{dx}{x^2+1}=1$$

$$c \arctan(x)\Big|_{-\infty}^{\infty}=1$$

$$c\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=1$$

$$c=1/\pi$$

b)

If $\frac{1}{3}\leq \tilde{X}^2\leq 1 \longrightarrow$ then $\frac{\sqrt{3}}{3}\leq \tilde{X}\leq 1$ or

$-1\leq \tilde{X}\leq -\frac{\sqrt{3}}{3}$

$$P\left(\frac{1}{3}\leq \tilde{X}^2\leq 1\right)=P\left(\frac{\sqrt{3}}{3}\leq \tilde{X}\leq 1\right)+P\left(-1\leq \tilde{X}\leq -\frac{\sqrt{3}}{3}\right)$$

$$P\left(\frac{\sqrt{3}}{3}\leq \tilde{X}\leq 1\right)=\int_{\sqrt{3}/3}^{1} f(x)\,dx$$

$$f(x)=\frac{1/\pi}{1+x^2}$$

$$P\left(-1\leq \tilde{X}\leq -\frac{\sqrt{3}}{3}\right)=\int_{-1}^{-\sqrt{3}/3} f(x)\,dx$$

$$=\int_{-1}^{-\sqrt{3}/3}\frac{1/\pi}{1+x^2}\,dx=\int_{\sqrt{3}/3}^{1}\frac{1/\pi}{1+x^2}\,dx$$

$\alpha=-x$

$d\alpha=-dx$

$$\int_{1}^{\sqrt{3}/3}\frac{1/\pi}{1+\alpha^2}(-d\alpha)$$

Note: $\int_a^b(\cdot)=-\int_b^a(\cdot)$

Thus

$$P\left(\frac{1}{3} \le X^2 \le 1\right) = \frac{2}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1}$$

$$= \frac{2}{\pi}\left[\tan^{-1}(1) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$$

$$= \frac{1}{6}$$

Ex:

X & Y are cont. RVs

Joint P.D.F. of X & Y is

$$f(x,y) = \begin{cases} cxy & 0<x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$$

a) $c = ?$ b) $P(1<X<2,\ 2<Y<3) = ?$

c) $P(X \geq 3,\ Y \leq 2) = ?$

Sln: $\int\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \longrightarrow \int_{1}^{5}\int_{0}^{4} cxy\,dx\,dy = 1$

a) $c\int_{1}^{5} x\,dx \int_{0}^{4} y\,dy = 1$

$$\boxed{c = 1/96}$$

(7)

b) $P(1<\tilde{x}<2,\ 2<\tilde{y}<3) = \int\limits_{x=0}^{4} \int\limits_{y=1}^{5} c\,x\,y\,dx\,dy$

Note

$$P(a<\tilde{x}<b,\ c<\tilde{y}<d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$P(1<\tilde{x}<2,\ 2<\tilde{y}<3) = \int\limits_{x=1}^{2} \int\limits_{y=2}^{3} \frac{1}{96}\,xy\,dx\,dy$$

$$= 96 \cdot \frac{1}{96} = 1$$

c) $P(\tilde{x} \geq 3,\ \tilde{y} \leq 2) = \int\limits_{x=3}^{4} \int\limits_{y=1}^{2} f(x,y)\,dy\,dx$

$$= \int\limits_{x=3}^{4} \int\limits_{y=1}^{2} \frac{1}{96}\,xy\,dy\,dx$$

$$= 7/128 \ //$$

Exe

$$f(x,y) = \begin{cases} \frac{xy}{96} & 0<x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$$

$$P(\tilde{x}+\tilde{y}<3) = ?$$

Sln: $P(\tilde{x}+\tilde{y}<3) = \iint\limits_{x+y<3} f(x,y)\,dx\,dy$

(8) $P(\tilde{x}+\tilde{y}<3) = \iint\limits_{x+y<3} f(x,y)\,dx\,dy$

$$f(x,y) = \begin{cases} \dfrac{xy}{96} & 0<x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$$

$$P(\tilde{x}+\tilde{y}<3) = \iint\limits_{x+y<3} f(x,y)\,dx\,dy$$

$$= \int_{x=0}^{4} \int_{y=1}^{3-x} \frac{xy}{96}\,dy\,dx = \frac{1}{48}$$

Since $x+y<3 \rightarrow y<3-x$

Ex: $P(2\tilde{x}+\tilde{y}<4) = ?$ $\quad f(x,y) = \begin{cases} k\,y & 0<x<3,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$

Sln: Solve it

Hint: find k $\quad 2x+y<4 \rightarrow y<4-2x$

$$\int_{x=0}^{3} \int_{y=1}^{4-2x} f(x,y)\,dx\,dy$$

Ex: $\tilde{x}, \tilde{y} \rightarrow$ RVs $\quad f(x,y) = \begin{cases} xy/96 & 0<x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$

$\tilde{u} = \tilde{x}^2\tilde{y} \quad \tilde{v} = \tilde{x}^2\tilde{y}$

$f(u,v) = ? \rightarrow$ Joint PDF of $\tilde{u}$ & $\tilde{v}$?

Use Jacobian matrix approach.

(9) **Ex:** $\tilde{X}$ & $\tilde{Y}$ are kont. RVs.

$$f(x,y) = \begin{cases} \frac{3}{4} + xy & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

a) $f(y|x) = ?$ b) $P(\tilde{Y} > \frac{1}{4} \mid \tilde{X} = \frac{1}{2}) = ?$

Sln:

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

a)

$$f(x) = \int_0^1 f(x,y)\,dy$$

$$f(x) = \int_0^1 \left(\frac{3}{4} + xy\right) dy$$

$$= \frac{3}{4} + \frac{x}{2}$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \begin{cases} \frac{3+4xy}{3+2x} & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

b) $P(\tilde{Y} > \frac{1}{2} \mid \tilde{X} = \frac{1}{2}) = \int_{1/2}^{1} f(y|\frac{1}{2})\,dy$

Note: $P(a < \tilde{Y} < b \mid \tilde{X} = x) = \int_a^b f(y|x)\,dy$

$P(\tilde{Y} \mid \tilde{X} = x) = \int_{-\infty}^{\infty} f(y|x)\,dy$

10

Thus, $p(\tilde{y}>\frac{1}{2}|\tilde{x}=\frac{1}{2})=\int_{1/2}^{1} f(y|\tfrac{1}{2})\,dy$

$$f(y|\tfrac{1}{2})=\begin{cases}\dfrac{3+4\cdot\frac{1}{2}\cdot y}{3+2\cdot\frac{1}{2}} & 0<y<1\\ 0 & \text{otherwise}\end{cases}$$

$$p(\tilde{y}>\tfrac{1}{2}|\tilde{x}=\tfrac{1}{2})=\int_{1/2}^{1}\frac{3+2y}{4}\,dy$$

$$=\frac{9}{16}$$

Ex:

$$f(x,y)=\begin{cases}8xy & 0\leq x\leq 1 \quad 0\leq y\leq x\\ 0 & \text{otherwise}\end{cases}$$

Joint p.d.f of $\tilde{x}$ & $\tilde{y}$

$f_{\tilde{x}}(x)=?$ $f_{\tilde{y}}(y)=?$ $f(x|y)=?$

$f(y|x)=?$

Ex:

$\tilde{u}=\tilde{x}\tilde{y}$ $f(x,y)\rightarrow$ Joint PDF of $\tilde{x}$ & $\tilde{y}$

pdf of $\tilde{u}$ in terms of $f(x,y)$?

Use Jacobian matrix

11

Exe

C.D.F. of cont. RV is given as

$$F(x)=\begin{cases} c x^3 & 0\le x<3 \\ 1 & x\ge 3 \\ 0 & \text{otherwise} \end{cases}$$

if $p(\tilde{x}=3)=0$

a) c? b) $p(\tilde{x}>1)$ c) $P(1<\tilde{x}<2)$

S/n:

$$p(a<\tilde{x}<b)=\int_a^b f(x)\,dx$$

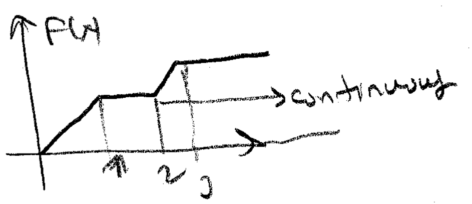
$$=F_{\tilde{x}}(b)-F_{\tilde{x}}(a)$$

$$p(\tilde{x}=a)=F_{\tilde{x}}(a)-F_{\tilde{x}}(a^-)$$

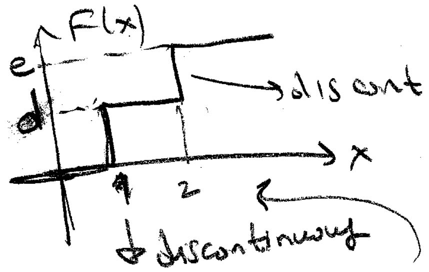
if $F(x)$ is continuous at $x=a$

then $p(\tilde{x}=a)=0$

For example



$p(\tilde{x}=2)=0$



$p(\tilde{x}=2)=e-d$

a) $p(\tilde{x}=3)=0$

$\Rightarrow F(3)-F(3^-)=0$

$1-c\,3^3=0 \rightarrow c=1/27$

b) $p(\tilde{x}>1)=P(1<\tilde{x}<\infty)=\underbrace{F(\infty)}_{=1}-F(1)=1-\frac{1}{27}1^3$

$=26/27$

12

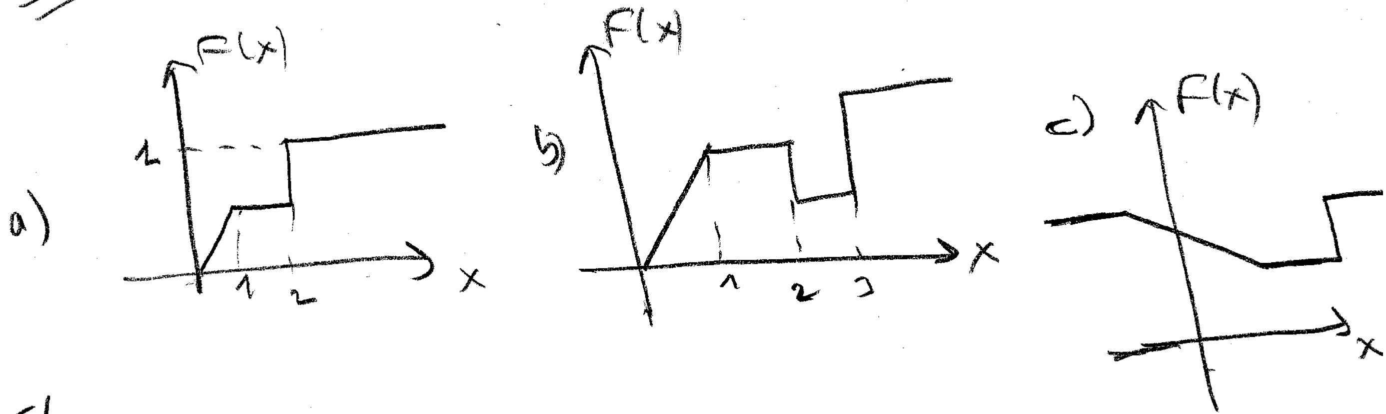
c) $P(1<\tilde{x}<2) = \int_1^2 f(x,y)\,dx\,dy$

$= F_{\tilde{x}}(2) - F_{\tilde{x}}(1)$

$= \frac{1}{27}\cdot 2^3 - \frac{1}{27} 1^3$

$= \frac{7}{27}$

Ex: Which one is a valid C.D.F function



Sln:

$F(x) \rightarrow$ monotonically increasing

$F(\infty)=1 \quad F(-\infty)=0$

Hence a) ✓   b) ✗   c) ✗

b) $\downarrow$ not valid $\downarrow$ decreases for $2 \leq x \leq 3$

c) $\downarrow$ not valid $F(-\infty) \neq 0$

13

Ex 3 Prove that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Sln: Let $\tilde{X} \sim N(0,1)$ (R.V.)

$\int_{-\infty}^{\infty} f_{\tilde{X}}(x) dx = 1 \qquad f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$\Rightarrow \int \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 \longrightarrow \int e^{-x^2/2} dx = \sqrt{2\pi}$

Let $x = \sqrt{2}u \rightarrow dx = \sqrt{2}\, du$

$\rightarrow \int_{-\infty}^{\infty} e^{-2u^2/2} \sqrt{2}\, du = \sqrt{2\pi}$

$\therefore \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$

Note: $\tilde{X} \sim N(m, \sigma^2)$

$P(a \le \tilde{X} \le b) = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)$

where $\Phi(x) = \int_{-\infty}^{x} e^{-u^2/2} du$

$\Phi(x)$ values are tabulated

(14)

Ex: $\hat{X} \sim N(2, 0.000064)$

$\hat{X} \to$ Gaussian R.V. with mean 8 and variance $\underbrace{0.000064}_{\sigma^2}$

Compute $p(1.98 \le \hat{X} \le 2.02)$

Sln: $p(1.98 \le \hat{X} \le 2.02) = \Phi\left(\dfrac{2.02 - 2.00}{\underbrace{0.008}_{\sigma}}\right)$

$$- \Phi\left(\frac{1.98 - 2.00}{0.008}\right)$$

$$= \underbrace{\Phi(2.5)}_{0.9938} - \underbrace{\Phi(-2.5)}_{(1-0.9938)}$$

from Tables in Books

$$= 0.9876$$

Note $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-u^2/2}\, du$

$$\Phi(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-u^2/2}\, du = -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{-u^2/2}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2}\, du$$

$$\Phi(x) + \Phi(-x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = \underbrace{\int f(x)\,dx}_{1}$$

Thus; $\boxed{\Phi(x) + \Phi(-x) = 1}$

$$\Phi(x) + \Phi(-x) = 1$$

Then For the previous example

$$p(1.98 \le \bar{X} \le 2.02) = \Phi(2.5) - \underbrace{\Phi(-2.5)}_{(1-\Phi(2.5))}$$

$$= 2 \times \Phi(2.5) - 1$$

$$= 2.(0.9938) - 1$$

$$= 0.9876$$

Exo

$\bar{X} \sim N(3, 0.0008)$

Compute $p(\bar{X} \ge 2) = ?$

$p(4 \le \bar{X} \le 5) = ?$

$p(2 \le \bar{X} \le 4) = ?$