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General Random Variables

Continuous Random Variables and PDFs

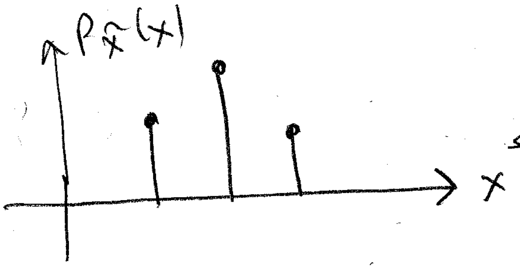
For discrete R.V. $\tilde{x}$

$$P(a \leq \tilde{x} \leq b) = \sum_{x=a}^{b} P_{\tilde{x}}(x)$$

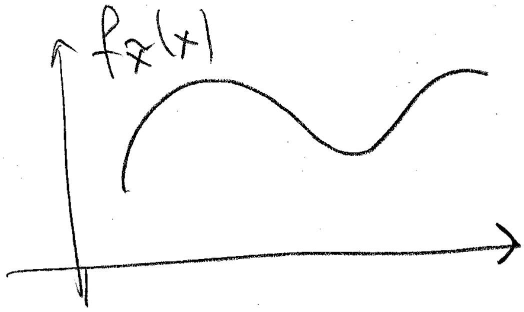
For continuous R.V. $\tilde{x}$

$$P(a \leq \tilde{x} \leq b) = \int_{a}^{b} f_{\tilde{x}}(x)\, dx$$

$f_{\tilde{x}}(x) \rightarrow$ Prob. density function of $\tilde{x}$



← discrete R.V. pdf



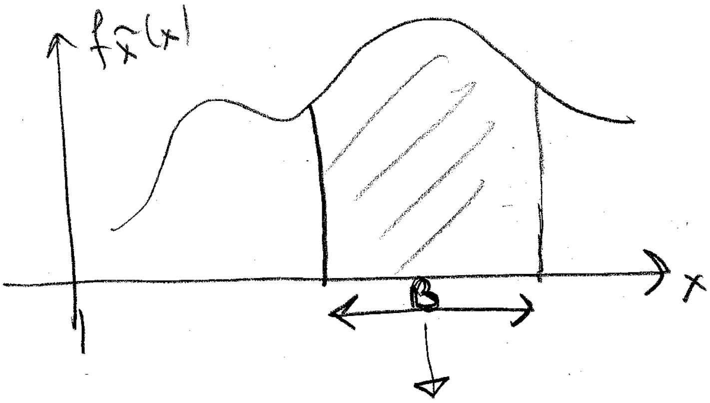
← continuous R.V. pdf

$x \leftarrow \tilde{x}$ can take x values

$\sum_{x} p(x) = 1 \rightarrow$ discrete R.V. case

$\int_{-\infty}^{+\infty} f(x)\, dx = 1 \rightarrow$ continuous R.V. case

②



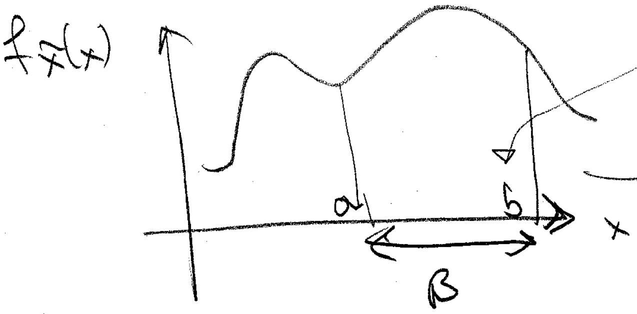
if all the real numbers are considered
then this part of the axis
is a subset of real line
and denoted by letter B

$x \rightarrow$ Values that $\tilde{X}$ R.V. can have.

$p(\tilde{X} \in B) = \text{Prob}(\tilde{X} \in B) \rightarrow$ The prob. of
R.V. taking values
from subset B
of the real line.

$$p(\tilde{X} \in B) = \int_B f_{\tilde{X}}(x)\,dx$$

$\rightarrow$ If frontiers are known
then



$$\rightarrow p(\tilde{X} \in B) = \int_a^b f_{\tilde{X}}(x)\,dx$$

For continuous R.V. $\tilde{x}$

$$P(\tilde{x}=a)=\int_a^a f(x)\,dx$$
$$=0$$

$$p(a\le\tilde{x}\le b)=p(a<\tilde{x}<b)=p(a\le\tilde{x}<b)=p(a<\tilde{x}\le b)$$

$$p(-\infty<\tilde{x}<\infty)=\int_{-\infty}^{\infty} f(x)\,dx=1$$

Note: For easy of notation $f(x)$ will be used for $f_{\tilde{x}}(x)$

Example for continuous R.V.

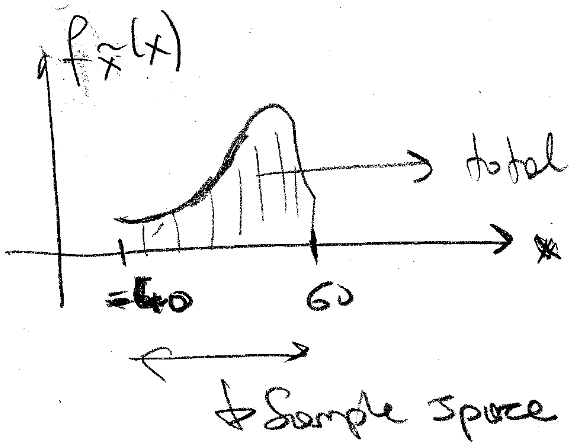
Weather temperature in 4 sessions

Sample space $S=\{-40 \dots 60\}$ → uncountable → Real Line

$\tilde{x}(s_i)=s_i$

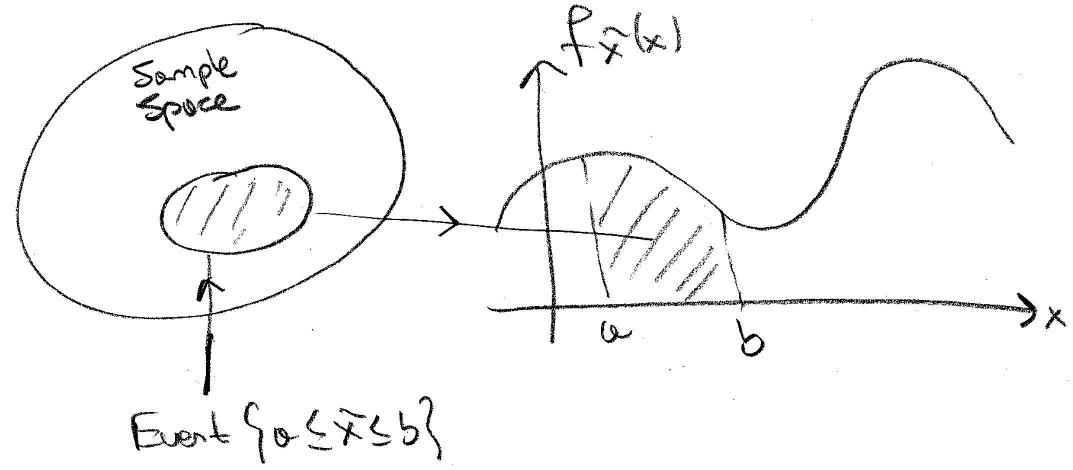
↓ ↓

infinitely many



→ total area = 1

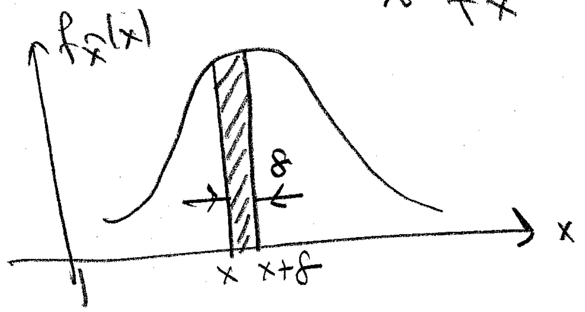
$$\int_{-40}^{60} f_{\tilde{x}}(x)\,dx=1$$



For an interval $[x, x+\delta]$ with very small δ we have,

$$P\left(\hat{X} \in [x, x+\delta]\right) = \int_{x}^{x+\delta} f_{\hat{X}}(x)\,dx$$

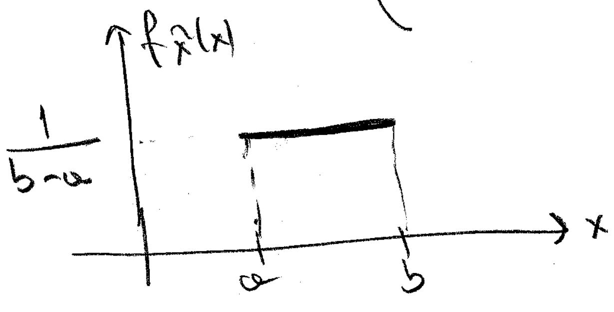
$$\approx f_{\hat{X}}(x)\,\delta$$



Ex: Continuous Uniform Random Variable

If $\hat{X} \rightarrow$ is a cont. unf. R.V.

then $f_{\hat{X}}(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b, \\ 0 & \text{otherwise;} \end{cases}$



$$\int_a^b f(x)\,dx = (b-a)\left(\frac{1}{b-a}\right)$$
$$= 1$$

Ex: $\tilde{X} \to$ R.V. with p.d.f.

$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx$$

$$= \sqrt{x}\,\Big|_0^1$$

$$= 1$$

Summary of PDF Properties:

$\tilde{X} \to$ cont. R.V. with p.d.f. $f_{\tilde{X}}(x)$ then

- $f_{\tilde{X}}(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f_{\tilde{X}}(x)\,dx = 1$
- if δ is very small $P(\tilde{X} \in [x, x+\delta]) = f_{\tilde{X}}(x)\,\delta$
- For any subset B of the real line

$$P(\tilde{X} \in B) = \int_B f_{\tilde{X}}(x)\,dx$$

Expectation and Variance

The expected or mean (average) of a cont. RV. $\tilde{X}$ is defined as

$$E(\tilde{X}) = \int_{-\infty}^{\infty} x\, f_{\tilde{X}}(x)\,dx$$

⑥ if $\tilde{y} = g(\tilde{x})$ then

$$E(g(\tilde{x})) = \int_{-\infty}^{\infty} g(x) f_{\tilde{x}}(x)\, dx$$

$$E(\tilde{x}^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$E(\tilde{x}^3) = \int_{-\infty}^{\infty} x^3 f(x)\, dx$$

$$E(\tilde{x}^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx$$

nth moment of cont. R.V. $\tilde{x}$

Variance of cont. R.V. $\tilde{x}$ is defined as

$$Var(\tilde{x}) = E\left[(\tilde{x} - E(\tilde{x}))^2\right]$$

$$= \int_{-\infty}^{\infty} (x - E(\tilde{x}))^2 f_{\tilde{x}}(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x - m)^2 f(x)\, dx$$

$m = E(\tilde{x})$ mean value

$Var(\tilde{x}) \geq 0$ and it can be showed that

$$Var(\tilde{x}) = E\left[(\tilde{x} - E[\tilde{x}])^2\right] = E(\tilde{x}^2) - [E(\tilde{x})]^2$$

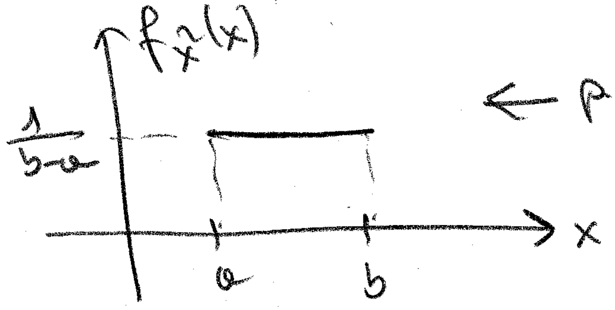
$$= E(\tilde{x}^2) - m^2$$

(7) **Ex:** $X \rightarrow$ uniform cont. R.V.

$E(X) = ?$

$Var(X) = ?$

Sln:



$\leftarrow$ p.d.f. of uniform R.V. X

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_a^b \frac{1}{(b-a)}\, x\, dx$$

$$= \frac{1}{(b-a)} \frac{x^2}{2} \Big|_a^b$$

$$= \frac{b+a}{2}$$

$$Var(X) = E\left[(X)^2\right] - \left[E(X)\right]^2$$

$$E(X^2) = \int_a^b x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \frac{(b^3 - a^3)}{3} = \frac{b^2 + ab + a^2}{3}$$

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$$\mathrm{var}(\tilde{x}) = E(\tilde{x}^2) - [E(\tilde{x})]^2$$

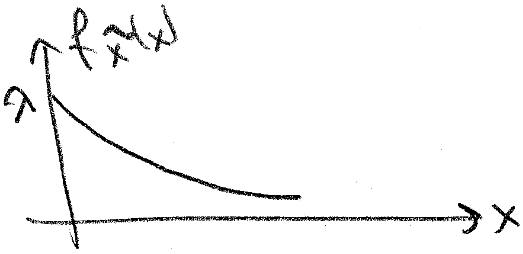
$$= \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

Exponential R.V. (Cont.)

An exponential R.V. has a p.d.f of the form

$$f_{\tilde{x}}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Notes For cont. R.V. $\tilde{x}$

$$p(a \leq \tilde{x} \leq b) = \int_a^b f(x)\,dx$$

$$p(\tilde{x} \geq b) = \int_b^\infty f(x)\,dx$$

$$p(\tilde{x} \leq b) = \int_{-\infty}^b f(x)\,dx$$

9 Ex:

$\tilde{x} \to$ Exponential R.V. (cont.)

$E(\tilde{x}) = ?$ $\quad$ Prob$(\tilde{x} \geq a) = ?$

$Var(\tilde{x}) = ?$

Sln:

For exp. R.V. $\tilde{x}$ $\quad f_{\tilde{x}}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$P(\tilde{x} \geq a) = \int_a^\infty f_{\tilde{x}}(x)\, dx$$

$$= \int_a^\infty \lambda e^{-\lambda x}\, dx$$

$$= -e^{-\lambda x}\Big|_a^\infty$$

$$= e^{-\lambda a}$$

$$E(\tilde{x}) = \int_{-\infty}^{\infty} x f_{\tilde{x}}(x)\, dx$$

$$= \int_0^\infty x.\lambda e^{-\lambda x} dx \qquad u = x \quad dv = \lambda e^{-\lambda x} dx$$

$$\qquad du = dx \quad v = -e^{-\lambda x}$$

$$= \int u\, dv$$

$$= uv - \int v\, du$$

$$= (-x e^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx$$

$$= 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$

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$$E(\tilde{X}^2) = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx \qquad u = x^2 \quad dv = \lambda e^{-\lambda x} dx$$

$$du = 2x \quad v = -e^{-\lambda x}$$

$$= \int u\, dv$$

$$= uv - \int v\, du$$

$$= \left(-x^2 e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x}\, dx$$

$$= 0 + \frac{2}{\lambda} E(\tilde{X})$$

$$= \frac{2}{\lambda^2}$$

$$Var(\tilde{X}) = E(\tilde{X}^2) - [E(\tilde{X})]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

Cumulative Distribution Functions (CDF)

If $\tilde{X}$ is discrete R.V.

c.d.f. of $\tilde{X}$ $F_{\tilde{X}}(x) = P(\tilde{X} \leq x)$

$$\underbrace{F_{\tilde{X}}(x)}_{\text{c.d.f. of } \tilde{X}} = \sum_{k \leq x} \underbrace{P_{\tilde{X}}(k)}_{\text{p.d.f. of } \tilde{X}}$$

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For cont. RV. $\tilde{X}$

$$F_{\tilde{X}}(x) = P(\tilde{X} \leq x)$$

$$= \int_{\infty}^{x} f_{\tilde{X}}(t)\, dt$$

p.d.f. of $\tilde{X}$

Hence;

$$F_{\tilde{X}}(x) = P(\tilde{X} \leq x) = \begin{cases} \sum_{k \leq x} P_{\tilde{X}}(k) & \tilde{X}\text{; discrete} \\ \int_{\infty}^{x} f_{\tilde{X}}(t)\,dt & \end{cases}$$

$F_{\tilde{X}}(x) \longrightarrow$ accumulates probability up to the value x

Properties of a CDF

$$F_{\tilde{X}}(x) = P(\tilde{X} \leq x) \quad \text{for all } x$$

- $F_{\tilde{X}}(x)$ is monotonically nondecreasing
  i.e., if $x \leq y$ then $F_{\tilde{X}}(x) \leq F_{\tilde{X}}(y)$
- $F_{\tilde{X}}(-\infty) = 0 \qquad F_{\tilde{X}}(+\infty) = 1$
- If $\tilde{X}$ is a discrete, then $F_{\tilde{X}}(x)$ has piecewise constant and staircase-like form
- If $\tilde{X}$ is cont, then $F_{\tilde{X}}(x)$ has a continuously varying form.

- If $\tilde{x}$ is discrete RV.

$$\underbrace{p_{\tilde{x}}(x_k)}_{\text{p.d.f. computed at } x = x_k} = F_{\tilde{x}}(x_k) - F_{\tilde{x}}(x_{k-1})$$

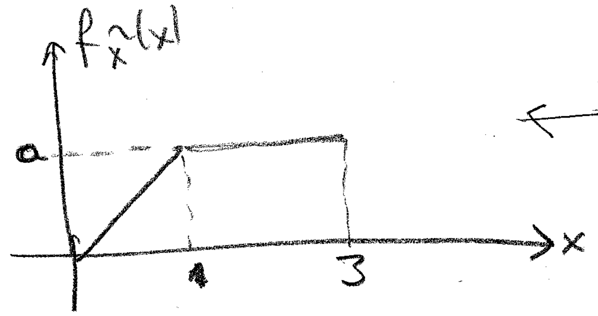
$$F_{\tilde{x}}(x) = \sum_{k=-\infty}^{x} p_{\tilde{x}}(k)$$

- if $\tilde{x}$ is cont. RV.

$$F_{\tilde{x}}(x) = \int_{-\infty}^{x} f_{\tilde{x}}(x)\,dx$$

$$f_{\tilde{x}}(x) = \frac{dF_{\tilde{x}}(x)}{dx}$$

(13) Ex:



$\leftarrow$ p.d.f. of cont. RV $\tilde{X}$

a) Find a b) Determine $F_{\tilde{X}}(x) \rightarrow$ c.d.f. of $\tilde{X}$

S/n:

$$\int_{\infty}^{\infty} f(x)\,dx = 1 \rightarrow \text{total area under } f_{\tilde{X}}(x) \text{ curve } = 1$$

$$\Rightarrow a \cdot \frac{1}{2} + 2a = 1 \longrightarrow \frac{5a}{2} = 1 \longrightarrow a = 2/5$$

$$F(x) = \int_{\infty}^{x} f(x)\,dx$$

when $0 \leq x < 1$ $f(x) = ax \rightarrow f(x) = \frac{2}{5}x$

$$F(x) = \int_{\infty}^{x} f(x)\,dx$$

$$= \int_{0}^{x} \frac{2}{5}x\,dx \rightarrow F(x) = \frac{2}{5}\frac{x^2}{2}$$

$$= \frac{x^2}{5}$$

when $1 \leq x < 3$

$$F(x) = \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

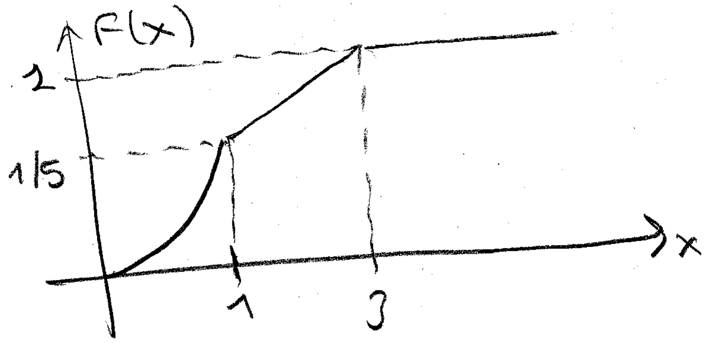
$$= \underbrace{\int_{0}^{1} ax\,dx}_{= a/2} + \int_{1}^{x} a\,dx$$

$$= \frac{1}{5} + \frac{2}{5}(x-1)$$

Graph of $F_{\tilde{X}}(x)$

$0 \leq x < 1 \quad F(x) = x^2/5$

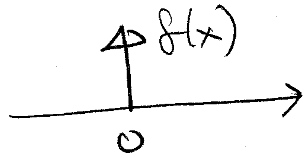
$1 \leq x < 3 \quad F(x) = \frac{1}{5} + \frac{2}{5}(x-1)$



Final value of F(x) equals 1

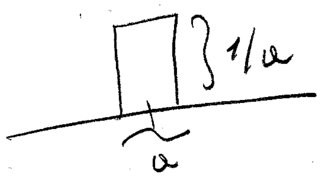
Impulse Functions

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$$



$\longrightarrow \delta(x) \quad a \to 0$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1 \longrightarrow$$

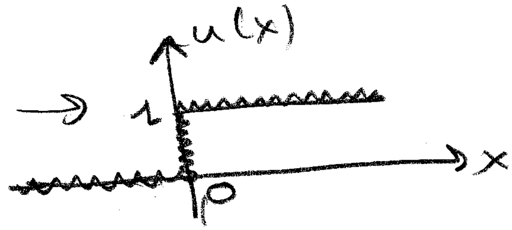


but total area of the rectangle $= 1$

$$\int_{-\infty}^{\infty} \delta(x - x_0)\,dx = 1$$

Unit Step Function

$$u(x) = \begin{cases} 1 & x > 0 \\ 0 & \text{otherwise} \end{cases} \rightarrow$$



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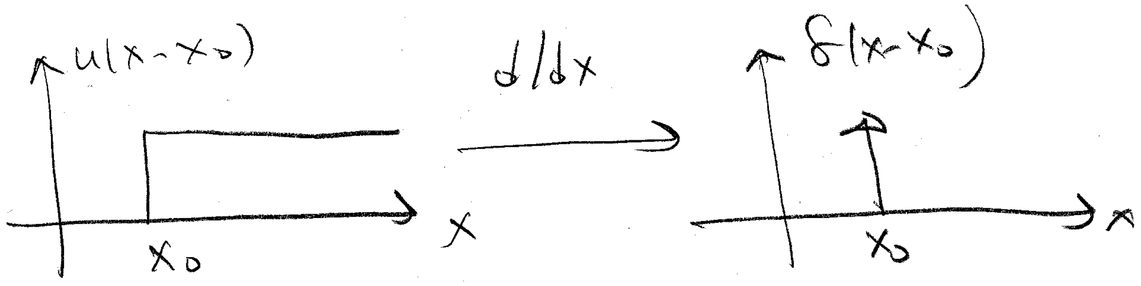
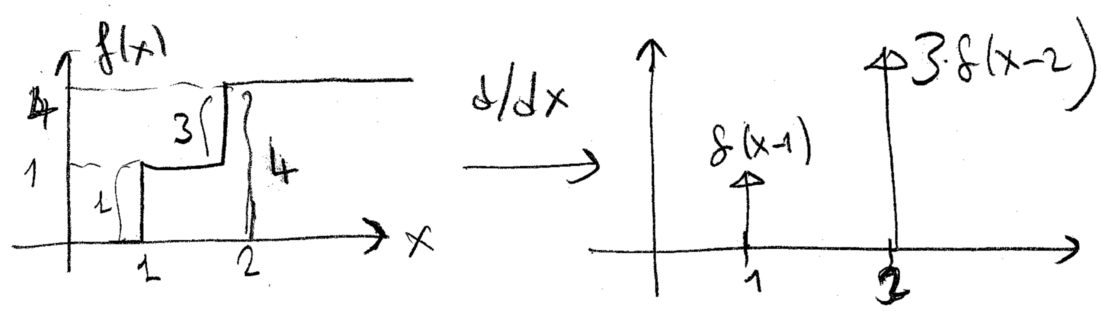
$$\delta(x) = \frac{d\,u(x)}{dx}$$

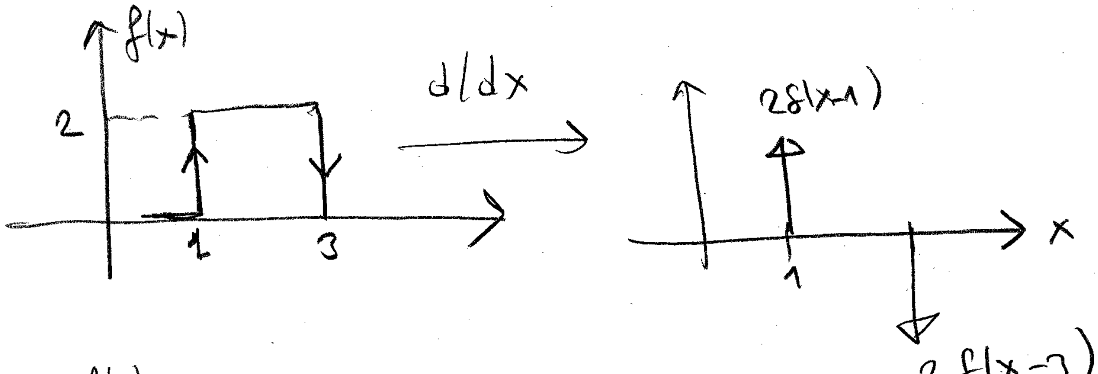
$$\delta(x-x_0) = \frac{d\,u(x-x_0)}{dx}$$



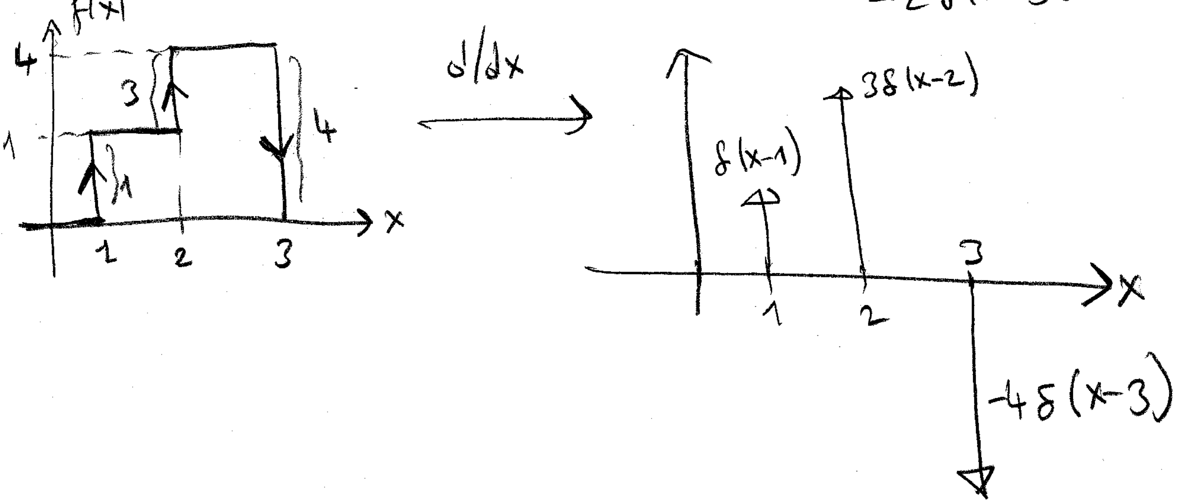
Εx:



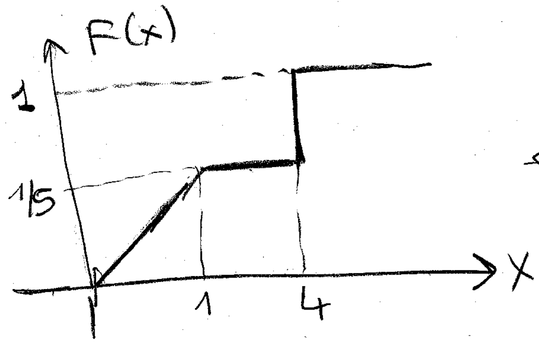
Εx:



Εx:



16 Ex:

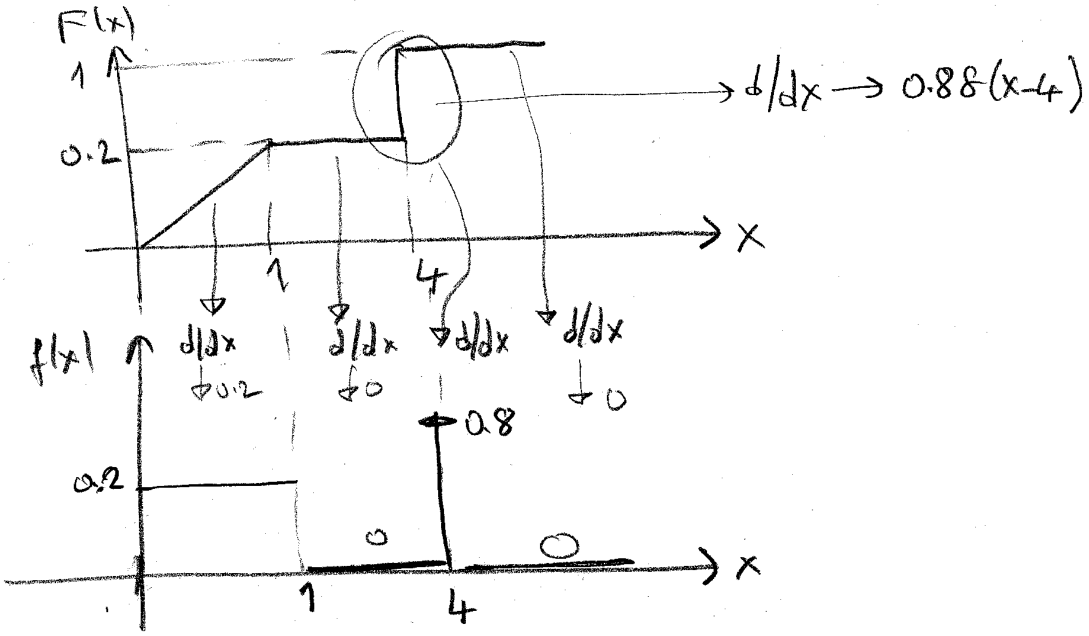


← c.d.f of cont R.V. $\tilde{x}$

find p.d.f. of $\tilde{x}$ $f_{\tilde{x}}(x)$

S/n:

$$f(x) = \frac{d\,F(x)}{dx}$$



Normal Random Variables:

A continuous random variable $\tilde{x}$ is said to be normal or Gaussian if it has a p.d.f. of the form

$$f_{\tilde{x}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}$$

(17)

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-m)^2/2\sigma^2}\,dx = 1$

$m \rightarrow$ is the mean of Gaussian R.V. $\tilde{X}$

$\sigma^2 \rightarrow$ is the variance of " " "

$m = \int_{-\infty}^{\infty} x\, f(x)\,dx \rightarrow m = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x\, e^{-(x-m)^2/2\sigma^2}\,dx$

$$\mathrm{Var}(\tilde{X}) = \sigma^2 = E\left((\tilde{X}-m)^2\right)$$

$$= \int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx$$

$$\Rightarrow \quad \sigma^2 = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} (x-m)^2\, e^{-(x-m)^2/2\sigma^2}\,dx$$

If $m=0$ & $\sigma=1$ then

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

$$\Rightarrow \overset{\sigma^2}{1} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx$$

$$\Rightarrow \sqrt{2\pi} = \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} \underbrace{x}_{u}\, \underbrace{x e^{-x^2/2}\,dx}_{dv} \rightarrow v = -e^{-x^2/2}$$

$\rightarrow$ integration by parts

$$= uv\Big|_{-\infty}^{\infty} - \int v\,du \rightarrow \underbrace{\left(-x e^{-x^2/2}\Big|_{-\infty}^{\infty}\right)}_{=0} + \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

(18) Hence, using integration by parts it is shown that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx = \int_{-\infty}^{\infty} e^{-x^2/2} dx$$

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-x^2/2} dx \rightarrow \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1}$$

<u>Property</u>

If $\tilde{x}$ is Gaussian R.V. with mean $m_{\tilde{x}}$ and variance σ^2

$$\tilde{y} = a\tilde{x} + b \rightarrow m_{\tilde{y}} = a\, m_{\tilde{x}}$$
$$\rightarrow \sigma^2_{\tilde{y}} = a^2 \sigma^2_{\tilde{x}}$$

<u>Note</u>: If $\tilde{x}$ is Gaussian with mean m & variance σ^2

$\tilde{x}$ is shown as $\tilde{x} \sim \underbrace{N}_{\text{means normal}}(\underbrace{m}_{\text{mean}}, \underbrace{\sigma^2}_{\text{variance}})$

<u>The Standard Normal R.V.</u>

A Gaussian (Normal) R.V. with zero mean and unit variance is said to be standard normal.

Its c.d.f. is computed as

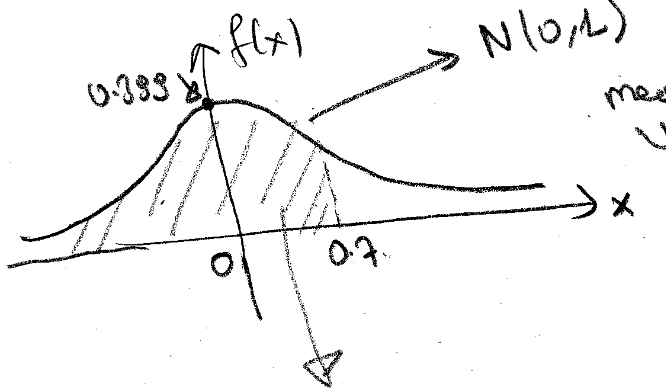
$$F_{\tilde{x}}(x) = P(\tilde{x} \leq x) = \int_{-\infty}^{x} f(x)\, dx$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

This specific c.d.f is denoted by the symbol $\Phi(x)$ → read as "phi"

Values of $\Phi(x)$ for different x are usually tabulated in books



mean = 0
variance = 1

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Area gives $\Phi(0.7)$ value

Ex:

$\tilde{y}$ is Gaussian (Normal) $\sim N(m, \sigma^2)$

$\tilde{y} \sim N(m, \sigma^2)$

$$\tilde{x} = \frac{\tilde{y} - m}{\sigma}$$

mean of $\tilde{x}$?
variance of $\tilde{x}$?

(20)

Sln:

$$E(\hat{y}) = E\left(\frac{\hat{X}-m}{\sigma}\right)$$

$$= \frac{\overbrace{E(\hat{X})}^{=m}}{\sigma} - \frac{m}{\sigma}$$

$$= \frac{m}{\sigma} - \frac{m}{\sigma}$$

$$= 0$$

$$Var(\hat{y}) = E\left[(\hat{y} - \underbrace{m_{\hat{y}}}_{0})^2\right]$$

$$= E\left[\hat{y}^2\right]$$

$$= E\left(\left(\frac{\hat{X}-m}{\sigma}\right)^2\right)$$

$$= \frac{1}{\sigma^2} E\left(\hat{X}^2 - 2m\hat{X} + m^2\right)$$

$$= \frac{1}{\sigma^2}\left[E(\hat{X}^2) - 2m\underbrace{E(\hat{X})}_{m} + m^2\right]$$

$$= \frac{1}{\sigma^2}\left[\underbrace{E(\hat{X}^2) - m^2}\right]$$

& Variance of $\hat{X}$ $= \sigma^2$

$$= \frac{\sigma^2}{\sigma^2}$$

$$= 1$$

Hence if $\tilde{y} \sim N(m, \sigma^2)$

then $\tilde{x} = \frac{\tilde{y}-m}{\sigma}$

$\tilde{x}$ is $N(0,1)$

c.d.f of $\tilde{y} \sim N(m, \sigma^2) = ?$

$$F_{\tilde{y}}(y) = P(\tilde{y} \leq y)$$

$$= P\Big(\underbrace{\frac{\tilde{y}-m}{\sigma}}_{\substack{\text{Normal} \\ \text{RV} \sim N(0,1)}} \leq \frac{y-m}{\sigma}\Big)$$

$$= \Phi\left(\frac{y-m}{\sigma}\right)$$

Lets change parameters:

If $\tilde{x} \sim N(m, \sigma^2)$

then $F_{\tilde{x}}(x) = \Phi\left(\frac{x-m}{\sigma}\right)$

c.d.f of $N(0,1)$ RV.