More than Two Random Variables

$\tilde{x}, \tilde{y}, \tilde{z}$ are RVs

$$p_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z) = \text{Prob}(\tilde{x}=x, \tilde{y}=y, \tilde{z}=z)$$
$$= \text{Prob}(\tilde{x}=x \cap \tilde{y}=y \cap \tilde{z}=z)$$

Marginal p.d.f's or p.m.f's are computed as

$$p_{\tilde{x},\tilde{y}}(x,y) = \sum_{z} p_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z)$$

$$p_{\tilde{x}}(x) = \sum_{y} \sum_{z} p_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z)$$

In a similar manner

$$p_{\tilde{y},\tilde{z}}(y,z) = \sum_{x} p_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z)$$

$$p_{\tilde{x},\tilde{z}}(x,z) = \sum_{y} p_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z)$$

$$p_{\tilde{y}}(y) = \sum_{x} \sum_{z} p_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z)$$

$$p_{\tilde{z}}(z) = \sum_{x} \sum_{y} p_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z)$$

③

$$P_{\hat{y},\hat{z}}(y,z) = \sum_{x} P_{\hat{x},\hat{y},\hat{z}}(x,y,z)$$

$$P_{\hat{y}}(y) = \sum_{z} \sum_{x} P_{\hat{x},\hat{y},\hat{z}}(x,y,z)$$

$$P_{\hat{z},\hat{y}}(z,y) = \sum_{x} P_{\hat{x},\hat{y},\hat{z}}(x,y,z)$$

$$P_{\hat{z}}(z) = \sum_{y} \sum_{x} P_{\hat{x},\hat{y},\hat{z}}(x,y,z)$$

$g(\hat{x},\hat{y},\hat{z}) \rightarrow$ function of RVs $\hat{x},\hat{y},$ & $\hat{z}$

$$E(g(\hat{x},\hat{y},\hat{z})) = \sum_{x,y,z} g(x,y,z)\, P_{\hat{x}\hat{y}\hat{z}}(x,y,z)$$

if $\hat{g} = a\hat{x} + b\hat{y} + c\hat{z} + d$

$$\Rightarrow E(\hat{g}) = a\,E(\hat{x}) + b\,E(\hat{y}) + c\,E(\hat{z}) + d$$

Note: if k is a constant number

$$E(k) = \sum k \cdot f(x)$$

$$= k \underbrace{\sum f(x)}_{1}$$

$$= k$$

④

Conditionings

The conditional p.d.f of RV $\hat{x}$ conditioned on a particular event A with $P(A) > 0$ is defined by

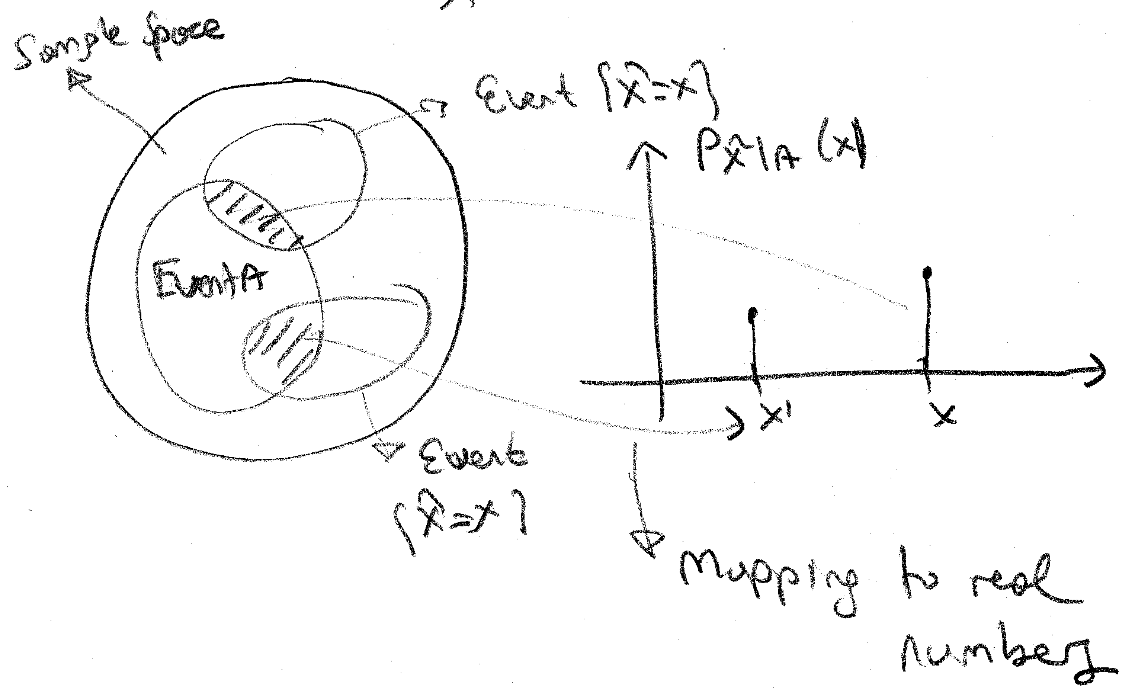
$$P_{\hat{x}|A}(x) = P(\hat{x} = x | A)$$

$$= \frac{P(\{\hat{x}=x\} \cap A)}{P(A)}$$

The events $\{\hat{x}=x\} \cap A$ are disjoint for different values of x, their union is A

therefore $P(A) = \sum_x P(\{\hat{x}=x\} \cap A)$

Hence $\sum_x P_{\hat{x}|A}(x) = 1$



5 Example

Experiment roll of a die

Sample Space $S=\{1,2,3,4,5,6\}$ → simple events

$\hat{X}(S_i)=S_i$ → Random variable defined on Sample space

$A=\{$ roll is even $\}$
$=\{2,4,6\}$

$$P_{\hat{X}|A}(x)=\frac{P(\hat{X}=x \cap A)}{P(A)}$$

$P(A)=1/2$ $\quad P_{\hat{X}|A}(1)=\frac{P(\{1\}\cap\{2,4,6\})}{P(A)}=0$

$$P_{\hat{X}|A}(2)=\frac{P(\{2\}\cap\{2,4,6\})}{1/2}$$

$$=\frac{1/6}{1/2}=\frac{1}{3}$$

In a similar manner

$P_{\hat{X}|A}(3)=0$

$P_{\hat{X}|A}(4)=1/3$ $\quad P_{\hat{X}|A}(5)=0$

$P_{\hat{X}|A}(6)=1/3$

$$P_{\hat{X}|A}(x)=\begin{cases}1/3 & \text{if } x=2,4,6.\\ 0 & \text{otherwise.}\end{cases}$$

(5) **Exe**

a coin is tossed two times

$S = \{hh, ht, th, tt\}$

$\hat{X}(s_i) = 2 \times$ number of heads in $s_i - 1$

$\hat{X}(hh) = 3 \quad \hat{X}(ht) = 1 \quad \hat{X}(th) = 1 \quad \hat{X}(tt) = -1$

$A = \{$at least one head$\} = \{hh, ht, th\}$

$P_{\hat{X}|A}(x) = ?$

S/n:

$P_{\hat{X}}(x) \rightarrow$ p.d.f. of $\hat{X}$

$P_{\hat{X}}(1) = 1/2 \quad P_{\hat{X}}(-1) = 1/4 \quad P_{\hat{X}}(3) = 1/4$

$$P_{\hat{X}|A}(x) = \frac{P(\{\hat{X} = x\} \cap A)}{P(A)} \qquad P(A) = \frac{3}{4}$$

$x = -1 \rightarrow P_{\hat{X}|A}(-1) = \dfrac{0}{P(A)} = 0$

$x = 1 \rightarrow P_{\hat{X}|A}(1) = \dfrac{P(\{\hat{X} = 1\} \cap A)}{P(A)} = \dfrac{P(\{ht, th\} \cap \{hh, ht, th\})}{P(A)}$

$= \dfrac{2/4}{3/4} = \dfrac{2}{3}$

$x = 3$

$\rightarrow P_{\hat{X}|A}(3) = \dfrac{1/4}{3/4} = 1/3$

$\sum_x P_{\hat{X}|A}(x) = \frac{2}{3} + \frac{1}{3} = 1$ ✓

(7) Conditioning one Random Variable on Another

$\hat{x}$ & $\hat{y}$ are two random variables associated with the same experiment

$$p_{\hat{x}|\hat{y}}(x|y) = P(\hat{x}=x\,|\,\hat{y}=y)$$

$$= \frac{P(\hat{x}=x,\ \hat{y}=y)}{P(\hat{y}=y)}$$

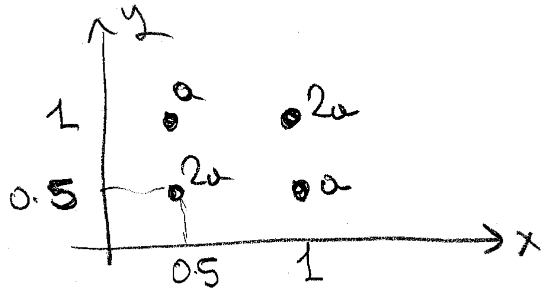
$$= \frac{p_{\hat{x},\hat{y}}(x,y)}{p_{\hat{y}}(y)}$$

$\sum_x p_{\hat{x}|\hat{y}}(x|y)=1$ since

$$\sum_x p_{\hat{x}|\hat{y}}(x|y) = \sum_x \frac{p_{\hat{x},\hat{y}}(x,y)}{p_{\hat{y}}(y)}$$

$p_{\hat{x},\hat{y}}(x,y)$ $$= \frac{p_{\hat{y}}(y)}{p_{\hat{y}}(y)} = 1$$

Ex:



$p_{\hat{x}|\hat{y}}(x|y) = ?$

Sln:

$$p_{\hat{x}|\hat{y}}(x|y) = \frac{p_{\hat{x},\hat{y}}(x,y)}{p_{\hat{y}}(y)}$$

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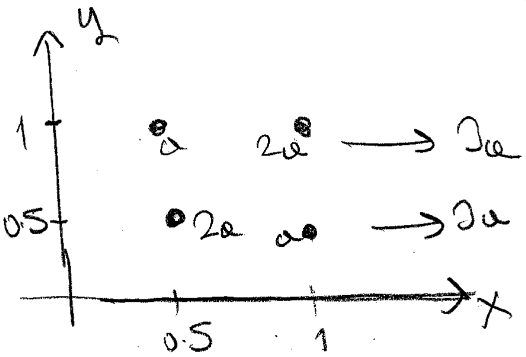
$$\sum p(x,y)=1 \rightarrow 6a=1 \rightarrow a=1/6$$



$$P_{\tilde{y}}(0.5)=3/6=1/2$$

$$P_{\tilde{y}}(1)=\frac{3}{6}=1/2$$

$$P_{\tilde{x}|\tilde{y}}(x|y)=\frac{P_{\tilde{x}\tilde{y}}(x,y)}{P_{\tilde{y}}(y)}$$

$y=0.5$ $\quad P_{\tilde{x}|\tilde{y}}(x|0.5)=\dfrac{P_{\tilde{x}\tilde{y}}(x,0.5)}{P_{\tilde{y}}(0.5)}$ $\nearrow x=0.5$, $\searrow x=1$

$y=1 \longrightarrow P_{\tilde{x}|\tilde{y}}(x|1)=\dfrac{P_{\tilde{x}\tilde{y}}(x,1)}{P_{\tilde{y}}(1)}$ $\nearrow x=0.5$, $\searrow x=1$

$y=0.5$
$\nearrow x=0.5 \longrightarrow P_{\tilde{x}|\tilde{y}}(0.5|0.5)=\dfrac{a}{3a}=1/3$
$\searrow x=1 \longrightarrow P_{\tilde{x}|\tilde{y}}(1|0.5)=\dfrac{2a}{3a}=2/3$
$+ = 1$

$y=1$
$\nearrow x=0.5 \longrightarrow P_{\tilde{x}|\tilde{y}}(0.5|1)=\dfrac{a}{3a}=1/3$
$\searrow x=1 \longrightarrow P_{\tilde{x}|\tilde{y}}(1|1)=\dfrac{2a}{3a}=2/3$
$+ = 1$

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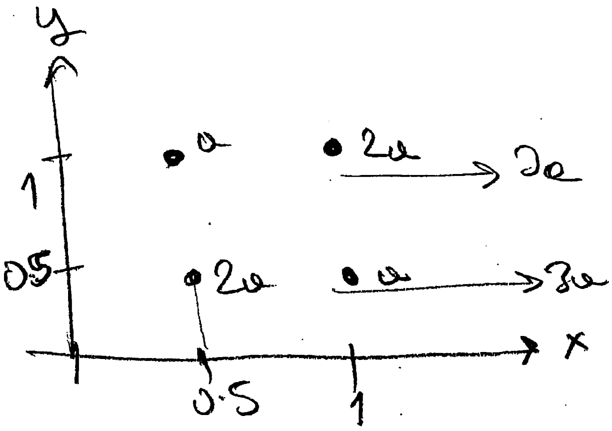
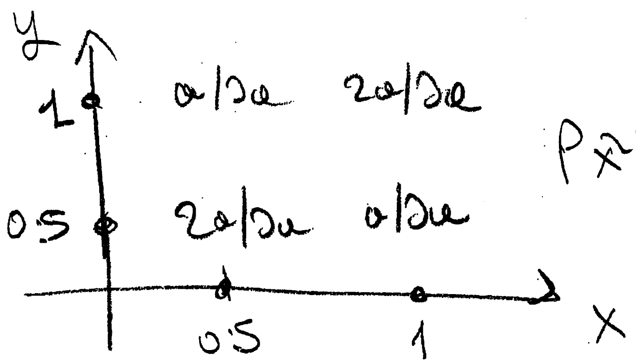


$p_{\tilde{y}}(y)$



$P_{\tilde{x}|\tilde{y}}(x|y)$

$$\sum_x P_{\tilde{x}|\tilde{y}}(x|y) = 1$$

Some properties:

$$P_{\tilde{x},\tilde{y}}(x,y) = P_{\tilde{y}}(y)\, P_{\tilde{x}|\tilde{y}}(x|y)$$

$$P_{\tilde{x},\tilde{y}}(x,y) = P_{\tilde{x}}(x)\, P_{\tilde{y}|\tilde{x}}(y|x)$$

$$P_{\tilde{x}}(x) = \sum_y P_{\tilde{x},\tilde{y}}(x,y)$$

$$= \sum_y P_{\tilde{y}}(y)\, P_{\tilde{x}|\tilde{y}}(x|y)$$

Note: $P_{\tilde{x}|A}(x) = P(\tilde{x} = x \mid A)$ $\quad \sum_x P_{\tilde{x}|A}(x) = 1$

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Conditional Expectations

$\tilde{x} \rightarrow$ R.V. $\quad A \rightarrow$ is an event

$\tilde{y} \rightarrow$ R.V. $\quad \tilde{x}, \tilde{y}, A \rightarrow$ associated with the same experiment

$$E(\tilde{x}|A) = \sum_x x \, p_{\tilde{x}|A}(x|A)$$

$$E(g(\tilde{x})|A) = \sum_x g(x) \, p_{\tilde{x}|A}(x|A)$$

$g(\tilde{x}) \rightarrow$ function of $\tilde{x}$

The conditional expectation of $\tilde{x}$ given a value y of $\tilde{y}$ is defined by

$$E(\tilde{x}|\tilde{y}=y) = \sum_x x \, p_{\tilde{x}|\tilde{y}}(x|y)$$

Since $p_{\tilde{x}}(x) = \sum_y p_{\tilde{y}}(y) \, p_{\tilde{x}|\tilde{y}}(x|y)$

$$\downarrow$$

$$E(\tilde{x}) = \sum_y p_{\tilde{y}}(y) \, E(\tilde{x}|\tilde{y}=y)$$

Total Expectation Theorem

$$E(\tilde{x}) = \sum_{i=1}^{n} P(A_i) E(\tilde{x}|A_i)$$

$A_1, A_2 .. A_n$ are disjoint events that form a partition of the sample space. $P(A_i) > 0$ for all i.

(11) Independence

Independence of a R.V. from an event

The R.V. $\tilde{X}$ is independent of the event A if

$$P(\tilde{X}=x \text{ and } A) = P(\tilde{X}=x \cap A)$$

$$= P(\tilde{X}=x)\,P(A)$$

$$= p_{\tilde{X}}(x)\,P(A) \quad \text{for all } x$$

$$p_{\tilde{X}|A}(x) = p_{\tilde{X}}(x) \quad \text{for all } x$$

<u>Independence of Random Variables:</u>

$\tilde{X}$ and $\tilde{Y}$ are independent ($\tilde{X}$, $\tilde{Y}$ are two RVs)

if

$$p_{\tilde{X},\tilde{Y}}(x,y) = p_{\tilde{X}}(x)\,p_{\tilde{Y}}(y) \quad \text{for all } x,y$$

$$p_{\tilde{X}|\tilde{Y}}(x|y) = p_{\tilde{X}}(x) \quad \text{for all } y \text{ with } p_{\tilde{Y}}(y) > 0$$

and all x.

$\tilde{X}$ & $\tilde{Y}$ are said to be conditionally independent, given a positive probability event A if

$$P(\tilde{X}=x, \tilde{Y}=y \mid A) = P(\tilde{X}=x \mid A)\,P(\tilde{Y}=y \mid A)$$

for all x and y

$$p_{\tilde{X},\tilde{Y}|A}(x,y) = p_{\tilde{X}|A}(x)\,p_{\tilde{Y}|A}(y)$$

for all x and y

(12) This is equivalent to

$$p_{\tilde{x}|\tilde{y},A}(x|y) = p_{\tilde{x}|A}(x) \quad \forall x, y \quad p_{\tilde{y}|A}(y) > 0$$

If $\tilde{x}$ and $\tilde{y}$ are independent random variables, then

$$E(\tilde{x}\tilde{y}) = E(\tilde{x})E(\tilde{y})$$

since;
$$E(\tilde{x}\tilde{y}) = \sum_x \sum_y xy\, p_{\tilde{x},\tilde{y}}(x,y)$$
$$= \sum_x \sum_y xy\, p_{\tilde{x}}(x)\, p_{\tilde{y}}(y)$$
$$= \sum_x x\, p_{\tilde{x}}(x) \sum_y y\, p_{\tilde{y}}(y)$$
$$= E(\tilde{x})\, E(\tilde{y})$$

If $\tilde{x}$ & $\tilde{y}$ are independent, then

$$E\big(g(\tilde{x})h(\tilde{y})\big) = E\big(g(\tilde{x})\big)\, E\big(h(\tilde{y})\big)$$

Ex://
$\tilde{x}$ & $\tilde{y}$ are two independent RVs

if $\tilde{z} = \tilde{x} + \tilde{y}$

show that $Var(\tilde{z}) = Var(\tilde{x}) + Var(\tilde{y})$

Sln// $Var(\tilde{z}) = E\left[(\tilde{z} - E(\tilde{z}))^2\right]$

$$= E\left[(\tilde{x} + \tilde{y} - E(\tilde{x} + \tilde{y}))^2\right]$$

(13)

Sln:

$$\mathrm{Var}(\tilde{z}) = E\left[(\tilde{x}+\tilde{y}-E(\tilde{x})-E(\tilde{y}))^2\right]$$

$$= E\left[(\tilde{x}+\tilde{y}-m_{\tilde{x}}-m_{\tilde{y}})^2\right]$$

$$= E\left[(\tilde{x}-m_{\tilde{x}}+\tilde{y}-m_{\tilde{y}})^2\right]$$

$$= E\left[(\tilde{x}-m_{\tilde{x}})^2+(\tilde{y}-m_{\tilde{y}})^2+2(\tilde{x}-m_{\tilde{x}})(\tilde{y}-m_{\tilde{y}})\right]$$

$$= E(\tilde{x}-m_{\tilde{x}})^2+E(\tilde{y}-m_{\tilde{y}})^2+2E\left[(\tilde{x}-m_{\tilde{x}})(\tilde{y}-m_{\tilde{y}})\right]$$

$$\to E\left[(\tilde{x}-m_{\tilde{x}})(\tilde{y}-m_{\tilde{y}})\right] = E\left[\tilde{x}\tilde{y}-m_{\tilde{y}}\tilde{x}-m_{\tilde{x}}\tilde{y}+m_{\tilde{x}}m_{\tilde{y}}\right]$$

$$= \overbrace{E(\tilde{x}\tilde{y})}^{=E(\tilde{x})E(\tilde{y})}-m_{\tilde{y}}E(\tilde{x})-m_{\tilde{x}}E(\tilde{y})+m_{\tilde{x}}m_{\tilde{y}}$$

$$= m_{\tilde{x}}m_{\tilde{y}}-m_{\tilde{y}}m_{\tilde{x}}-m_{\tilde{x}}m_{\tilde{y}}+m_{\tilde{x}}m_{\tilde{y}}$$

$$= 0$$

Hence,

$$\mathrm{Var}(\tilde{z}) = E\left[(\tilde{x}-m_{\tilde{x}})^2\right] + E\left[(\tilde{y}-m_{\tilde{y}})^2\right]$$

$$= \mathrm{Var}(\tilde{x})+\mathrm{Var}(\tilde{y})$$

Summary:

A is an event with $P(A) > 0$

$\tilde{x}$ & $\tilde{y}$ are two RVs associated with the same experiment

- $\tilde{x}$ is independent of the event A if

$$p_{\tilde{x}|A}(x) = p_{\tilde{x}}(x) \quad \forall x$$

- if $\tilde{x}$ & $\tilde{y}$ are independent RVs then

$$\underbrace{p_{\tilde{x},\tilde{y}}(x,y)}_{\text{Joint p.d.f of } \tilde{x} \& \tilde{y}} = \underbrace{p_{\tilde{x}}(x)}_{\text{Marginal pdf for } \tilde{x}} \underbrace{p_{\tilde{y}}(y)}_{\text{Marginal p.d.f for } \tilde{y}}$$

- If $\tilde{x}$ & $\tilde{y}$ are independent RVs then $E(\tilde{x}\tilde{y}) = E(\tilde{x})E(\tilde{y})$

Furthermore, for any functions f and g, the random variables $f(\tilde{x})$ & $g(\tilde{y})$ are independent,

$$E\left(f(\tilde{x})\,g(\tilde{y})\right) = E\left(f(\tilde{x})\right) E\left(g(\tilde{y})\right)$$

- If $\tilde{x}$ & $\tilde{y}$ are independent, then

$$Var(\tilde{x} + \tilde{y}) = Var(\tilde{x}) + Var(\tilde{y})$$

Independence of Several RVs

Three random variables $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are said to be independent if $p_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z) = p_{\tilde{x}}(x)\, p_{\tilde{y}}(y)\, p_{\tilde{z}}(z)$ for all x, y, z

(15) If $\tilde{x}, \tilde{y}$, and $\tilde{z}$ are independent RVs then $f(\tilde{x}), g(\tilde{y}), h(\tilde{z})$ are also independent

- $g(\tilde{x}, \tilde{y})$, $h(\tilde{z})$ are independent
- $f(\tilde{x}, \tilde{y}), g(\tilde{y}, \tilde{z})$ are not independent due to common use of $\tilde{y}$ in both f and g functions.

For the three RVs $\tilde{x}, \tilde{y}, \tilde{z}$ the joint p.d.f.

$$P_{\tilde{x},\tilde{y},\tilde{z}}(x,y,z) = P_{\tilde{y}}(y)\, P_{\tilde{y}|\tilde{z}}(y|z)\, P_{\tilde{x}|\tilde{y},\tilde{z}}(x|y,z)$$

<u>Examples:</u>

Ex: Suppose that n people throw their hats in a box and then picks up one hat at random. What is the expected value of $\tilde{x}$, the number of people that get back their own hat?

Sln: For the ith person we introduce a RU $\tilde{x}_i$ that takes the value 1 if the person selects his/her own hat, and takes the value 0 otherwise.

$$P(\tilde{x}_i = 1) = \frac{1}{n} \quad \& \quad P(\tilde{x}_i = 0) = 1 - \frac{1}{n}$$

the mean of $\tilde{x}_i$ is $E(\tilde{x}_i) = 1 \cdot \frac{1}{n} + 0\left(1 - \frac{1}{n}\right)$

$$= \frac{1}{n}$$

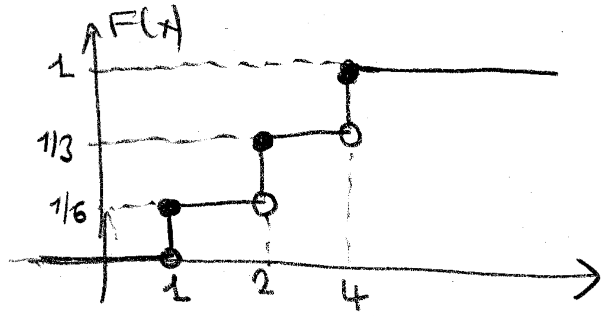
(16) We now have $\tilde{X} = \tilde{X}_1 + \tilde{X}_2 + \dots + \tilde{X}_n$

$$E(\tilde{X}) = E(\tilde{X}_1) + E(\tilde{X}_2) + \dots + E(\tilde{X}_n)$$

$$= n \cdot \frac{1}{n}$$

$$= 1$$

Ex 3



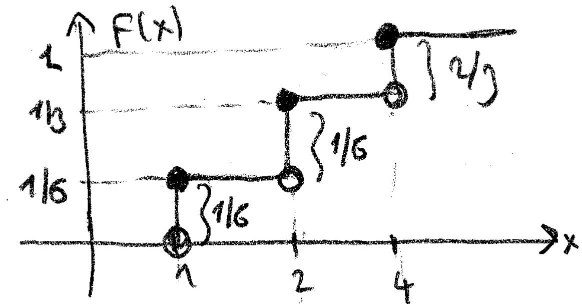
← c.d.f. of RV $\tilde{X}$ is given

a) $P_{\tilde{X}}(x) = ?$ → p.d.f. of $\tilde{X}$

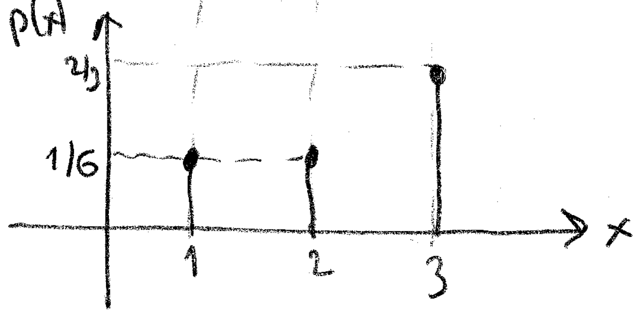
b) $E(\tilde{X}) = ?$ c) $Var(\tilde{X}) = ?$

Sln:

$$P_{\tilde{X}}(x_i) = F(x_i) - F(x_{i-1})$$



a)



b) $E(\tilde{X}) = \sum x\, p(x)$

$$= (1)(\tfrac{1}{6}) + (2)(\tfrac{1}{6}) + 3 \cdot \tfrac{2}{3}$$

$$= \frac{3}{6} + 2$$

$$= \frac{1}{2} + 2$$

$$= 5/2 = 2.5$$

(17)

c) $Var(\tilde{X}) = E\left[(\tilde{X} - m_{\tilde{X}})^2\right]$

$= E\left[(\tilde{X} - 2.5)^2\right]$

$= E((\tilde{X})^2) - m_{\tilde{X}}^2$

$= E(\tilde{X}^2) - 2.5^2$

$= E(\tilde{X}^2) - 6.25$

$E(\tilde{X}^2) = \sum x^2 p(x)$

$= 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{2}{3}$

$= \frac{5}{6} + 6$

$= \frac{41}{6}$

$Var(\tilde{X}) = \frac{41}{6} - 6.25$

$=$