1

Ex:

$\tilde{y} = |\tilde{x}|$ $\tilde{x}, \tilde{y}$ are two random variables.

p.d.f of $\tilde{x}$ is given as

$$P_{\tilde{x}}(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in the range } [-4, 4]; \\ 0 & \text{otherwise} \end{cases}$$

Determine p.d.f. of $\tilde{x}$, i.e, $P_{\tilde{y}}(y) = ?$

Sln:

$$P_{\tilde{y}}(y) = \sum_{\{x \mid g(x) = y\}} P_{\tilde{x}}(x) \qquad \tilde{y} = g(\tilde{x})$$

$\tilde{y} \rightarrow$ takes only nonnegative values, i.e, 0,1,2,3,4

We should determine x values such that they satisfy $\{x \mid g(x) = y\}$ for each y value

$y \rightarrow$ takes values 0,1,2,3,4

For

$y=0 \rightarrow y=g(x) \rightarrow 0 = g(x) \rightarrow x=0$

$y=1 \rightarrow y=g(x) \rightarrow 1=|x| \rightarrow x=\pm 1;$

$y=2 \rightarrow y=g(x) \rightarrow 2=|x| \rightarrow x=\pm 2;$

$y=3 \rightarrow y=g(x) \rightarrow 3=|x| \rightarrow x=\pm 3;$

$y=4 \rightarrow y=g(x) \rightarrow 4=|x| \rightarrow x=\pm 4;$

Since $P_{\tilde{y}}(y) = \sum_{\{x \mid g(x) = y\}} P_{\tilde{x}}(x) \rightarrow$

$P_{\tilde{y}}(0) = P_{\tilde{x}}(0)$

$P_{\tilde{y}}(1) = P_{\tilde{x}}(-1) + P_{\tilde{x}}(1)$

$P_{\tilde{y}}(2) = P_{\tilde{x}}(-2) + P_{\tilde{x}}(2)$

$P_{\tilde{y}}(3) = P_{\tilde{x}}(-3) + P_{\tilde{x}}(3)$

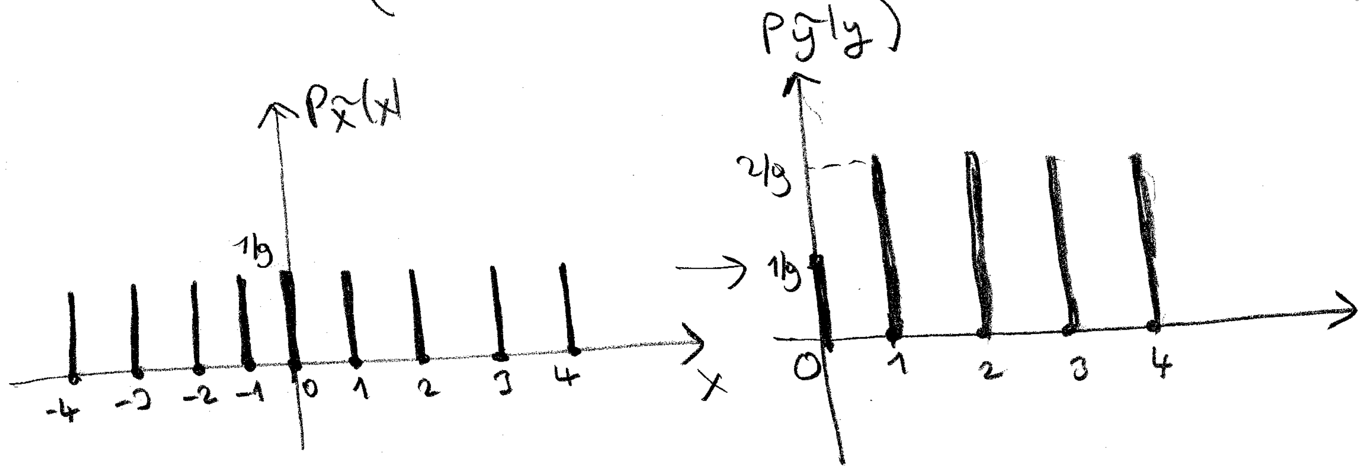
(2)

$$P_{\tilde y}(4) = P_{\tilde x}(-4) + P_{\tilde x}(4)$$

Hence $P_{\tilde y}(0) = 1/9$

$$P_{\tilde y}(1) = 2/9$$
$$P_{\tilde y}(2) = 2/9$$
$$P_{\tilde y}(3) = 2/9$$
$$P_{\tilde y}(4) = 2/9$$

$$P_{\tilde y}(y) = \begin{cases} 2/9 & \text{if } y = 1, 2, 3, 4 \\ 1/9 & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$



Exo

Mean and Variance of the Bernoulli

$\tilde X \to$ Bernoulli R.V.

For instance: tossing a biased coin

Prob(h) = p   Prob(t) = 1 − p $\to$ $\tilde X(h) = 1$ $\tilde X(t) = 0$

p.d.f for $\tilde X$: $P_{\tilde X}(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$

Mean value of Bernoulli R.V.

$$E(\tilde{X}) = \sum x \, p(x)$$
$$= 1.p + 0.(1-p) = p$$

Variance of Bernoulli R.V.

$$Var(\tilde{X}) = E(\tilde{X}^2) - \left[E(\tilde{X})\right]^2$$

$$E(\tilde{X}^2) = \sum x^2 p(x)$$
$$= 1^2 p + 0^2(1-p) = p$$

$$Var(\tilde{X}) = p - (p^2)$$
$$= p(1-p)$$
$$= p.q$$

Discrete Uniform Distributed Random Variable

$\tilde{X} \rightarrow$ discrete uniform distributed R.V.

p.d.f. of $\tilde{X}$ is defined of

$$p_{\tilde{X}}(x) = \begin{cases} \dfrac{1}{b-a+1} & \text{if } x = a, a+1, \ldots b \\ 0 & \text{otherwise} \end{cases}$$

a, b are two integers such that $a < b$

(4)

Ex: $\tilde{X} \rightarrow$ uniform distributed R.V.

find mean and variance of $\tilde{X}$

Sln:

$$p_{\tilde{X}}(x) = p(x) = \begin{cases} \frac{1}{b-a+1} & \text{if } k = a, a+1, \dots b \\ 0 & \text{otherwise,} \end{cases}$$

$$E(\tilde{X}) = m_{\tilde{X}} = \sum_{x=a}^{b} x p(x)$$

$$= \frac{a}{b-a+1} + \frac{a+1}{b-a+1} + \dots \frac{\overbrace{b}^{=a+m}}{b-a+1}$$

$$= \frac{a + a+1 + \dots a+m}{b-a+1}$$

$$= \frac{\overbrace{a+a+\dots+a}^{m+1 \text{ terms}} + 1+2+\dots m}{b-a+1}$$

$$= \frac{(m+1)a + m(m+1)/2}{b-a+1}$$

$$= \frac{(m+1)(a+m/2)}{b-a+1}$$

put $m = b-a$

$$= \frac{\cancel{(b-a+1)}\left(\frac{a+b}{2}\right)}{\cancel{b-a+1}} = \frac{a+b}{2}$$

(5)

Variance of $\tilde{x}$ $Var(\tilde{x}) = E(\tilde{x}^2) - [E(\tilde{x})]^2$

$$E(\tilde{x}^2) = \sum_{x=a}^{b} x^2 p(x)$$

$$= a^2 p(a) + \dots b^2 p(a)$$

$$= \frac{a^2 + \dots b^2}{b-a+1}$$

For $a=1$ $b=m$ $\longrightarrow m = b-a+1$

$$E(\tilde{x}^2) = \frac{1}{6}(m+1)(2m+1)$$

$$E(\tilde{x}) = \frac{a+b}{2} = \frac{1+m}{2}$$

$$Var(\tilde{x}) = E(\tilde{x}^2) - [E(\tilde{x})]^2$$

$$= \frac{1}{6}(m+1)(2m+1) - \frac{1}{4}(m+1)^2$$

$$= \frac{m^2-1}{12}$$ $\leftarrow$ put $m = b-a+1$

⑥

taking $m = b - a + 1$

$$Var(X) = \frac{(b-a+1)^2 + 1}{2}$$

$$= \frac{(b-a)(b-a+2)}{12}$$

Example: The roll of a four sided die

R.V. defined of this experiment has the following p.d.f.

$$p_X(x) = \begin{cases} 1/6 & \text{if } x = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

Variance of X?

Sln:

$$E(X) = \sum x\, p(x)$$

$$= \sum_{x=1}^{6} x \cdot \frac{1}{6} = 3.5$$

$$E(X^2) = \sum x^2 p(x)$$

$$= \sum_{x=1}^{6} x^2 p(x)$$

$$= \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 35/12$$

(7) Ex8 Mean of Poisson R.V.

$\tilde{x} \to$ Poisson R.V. then $\tilde{x}$ has

p.d.f. $p_{\tilde{x}}(x) = e^{-\lambda} \dfrac{\lambda^{x}}{x!}$ $x = 0, 1, 2, ..$

$x \to$ integer

$$E(\tilde{x}) = \sum_{x=0}^{\infty} x p(x)$$

$$= \sum_{x=0}^{\infty} x \, e^{-\lambda} \frac{\lambda^{x}}{x!}$$

$$= \lambda \sum_{x=1}^{\infty} e^{-\lambda} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda \underbrace{\sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^{m}}{m!}} \quad m = x - 1$$

$$= \lambda$$

$$e^{-\lambda} \underbrace{\sum_{m=0}^{\infty} \frac{\lambda^{m}}{m!}}_{e^{\lambda}}$$

$$e^{-\lambda} e^{\lambda} = 1$$

8

Joint Prob. Density Function of Multiple Random Variables

$\hat{x}, \hat{y}$ are two RVs

Joint p.d.f of $\hat{x}$ & $\hat{y}$ is defined as

$$P_{\hat{x},\hat{y}}(x,y) = \text{Prob}(\hat{x}=x, \hat{y}=y)$$

$(\hat{x}=x, \hat{y}=y) \rightarrow$ means $\rightarrow \{s_i \mid \hat{x}(s_i)=x\} \cap \{s_i \mid \hat{y}(s_i)=y\}$

is a set

intersection

$$(\hat{x}=x, \hat{y}=y) = (\hat{x}=x) \cap (\hat{y}=y)$$

Notations

capital p

$$\text{Prob}(\hat{x}=x, \hat{y}=y) = P(\hat{x}=x, \hat{y}=y)$$

$$= P(\{\hat{x}=x\} \cap \{\hat{y}=y\})$$

$$= P(\hat{x}=x \text{ and } \hat{y}=y)$$

$$= P(\{s_i \mid \hat{x}(s_i)=x\} \cap \{s_i \mid \hat{y}(s_i)=y\})$$

all are the same

⑨

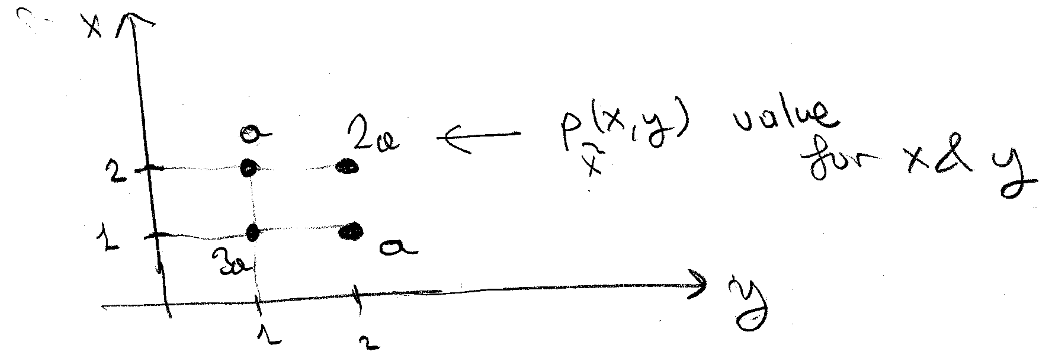
$p_{\tilde{x},\tilde{y}}(x,y) \rightarrow$ Joint p.d.f. of $\tilde{x}$ & $\tilde{y}$

Marginal p.d.f.s $p_{\tilde{x}}(x)$ & $p_{\tilde{y}}(y)$ are computed from $p(x,y)$ as

$$p_{\tilde{x}}(x) = \sum_{x} p_{\tilde{x},\tilde{y}}(x,y)$$

$$p_{\tilde{y}}(y) = \sum_{y} p_{\tilde{x},\tilde{y}}(x,y)$$

Ex: Joint p.d.f. of $\tilde{x}$ & $\tilde{y}$ is shown below



a) Determine a?

b) Determine $p(x)$ c) Determine $p(y)$

Sln: $\sum p(x,y) = 1 \rightarrow a + 2a + 3a + a = 1$

$7a = 1 \rightarrow a = 1/7$

10

b)

$$p_X(x) = \sum_y p(x,y)$$

$$p_X(x) = \sum_{y=1}^{2} p(x,y)$$

$$p_X(x) = p(x,1) + p(x,2)$$

for $x=1 \rightarrow p_X(1) = p(1,1) + p(1,2)$

$= 3a + a$

$= 4/7$

for $x=2$ $\quad p_X(2) = p(2,1) + p(2,2)$

$= a + 2a$

$= 3/7$

c) $\quad p_Y(y) = \sum_x p(x,y)$

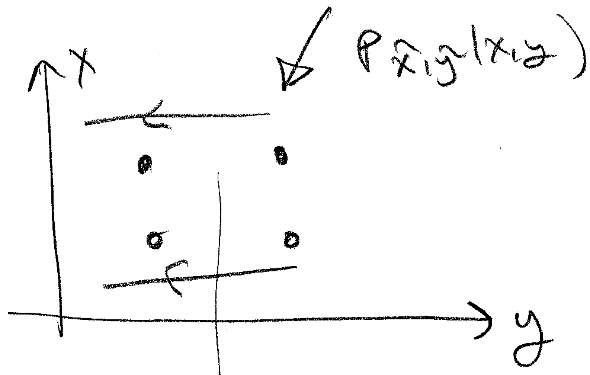
$p_Y(y) = p(1,y) + p(2,y)$

$y=1 \rightarrow p_Y(1) = p(1,1) + p(2,1)$

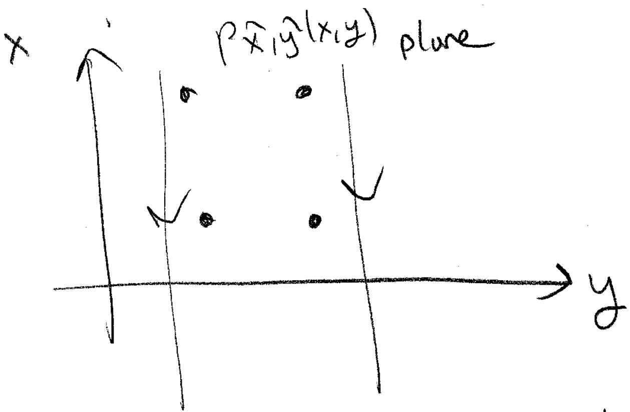
$= a + 3a$

$= 4/7$

$y=2$ $\quad p_Y(2) = p(1,2) + p(2,2) \rightarrow p_Y(2) = 3/7$

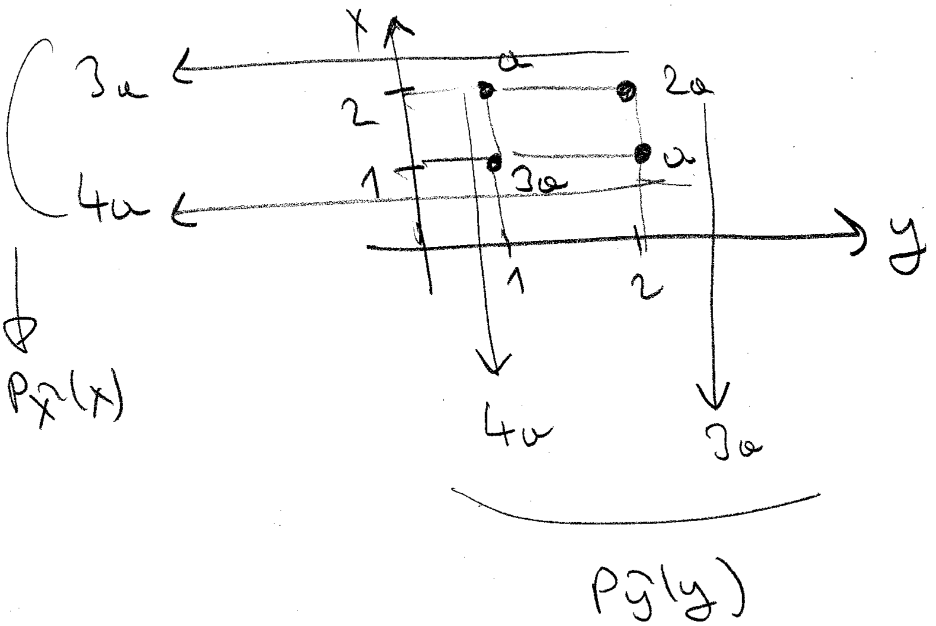


sum them in this direction
to get $p_{\hat{x}}(x)$



sum them in this direction to get $p_{\hat{y}}(y)$

For our example



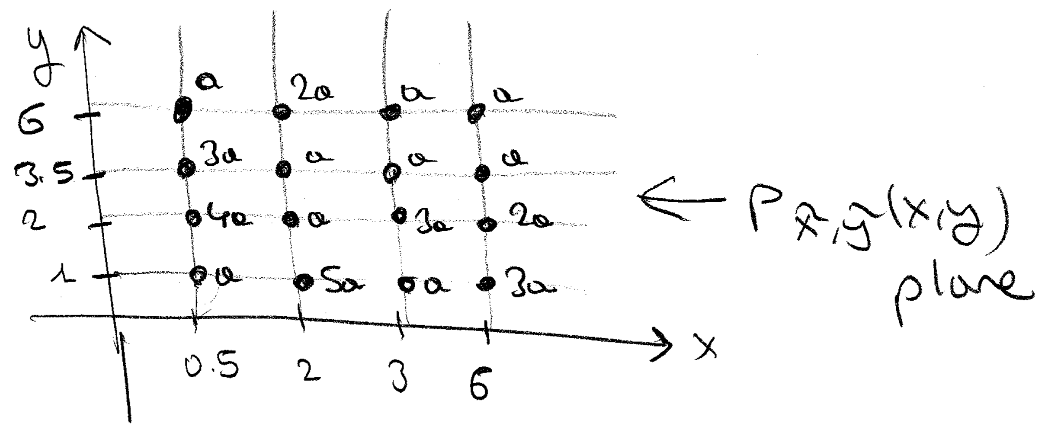
Note $\sum p(x) = 7a = 1$ $\quad$ $\sum p(y) = 7a = 1$

(12)

Ex:

$$p_{\tilde{x},\tilde{y}}(x,y) \longrightarrow \overset{\text{Joint}}{\text{p.d.f.}} \text{ of } \tilde{x} \,\&\, \tilde{y}$$

Plane of $p_{\tilde{x},\tilde{y}}(x,y)$ is given below



a) Determine a

b) Determine $p_{\tilde{x}}(x)$ & $p_{\tilde{y}}(y)$

c) Determine $\text{Prob}(\tilde{x} \le 3, \tilde{y} \le 6)$