(1) Experiment: Toss of a fair and biased coin

For a fair coin $p(h)=1/2$ $p(t)=1/2$

For a biased coin $p(h_b)=2/3$ $p(t_b)=1/3$

Sample space $S=\{hh_b, ht_b, th_b, tt_b\}$

$p(x)=?$ prob. mass function

$F(x)=?$ cumulative distribution function

$\hat{X}(s_i)=\{2\times$ number of heads $+ 3\times$ number of biased tails$\}$

S/n:

$\hat{X}(hh_b)=2\times2+3\times0 \rightarrow \hat{X}(hh_b)=4$

$\hat{X}(ht_b)=2\times1+3\times1 \rightarrow \hat{X}(ht_b)=5$

$\hat{X}(th_b)=2\times0\ 3\times0 \rightarrow \hat{X}(th_b)=0$

$\hat{X}(tt_b)=2\times0+3\times1 \rightarrow \hat{X}(tt_b)=3$

$p(x)=Prob(\hat{X}=x) \rightarrow x$ can take values 0, 3, 4, 5

$x=0$ $p(x)=Prob(\hat{X}=0) \rightarrow p(0)=Prob(\hat{X}=0)$

$=Prob(th_b)$

$=\frac{1}{2}\cdot\frac{2}{3}$

$=1/3$

$x=3$ $p(x)=Prob(\hat{X}=x)$

$\rightarrow p(3)=Prob\{tt_b\}$

$=\frac{1}{2}\cdot\frac{1}{3} \rightarrow 1/6 \rightarrow p(3)=1/6$

$p(4)=Prob(\hat{X}=4)$

$=Prob(hh_b)$

$=\frac{1}{2}\cdot\frac{2}{3}$

$=1/3$

$p(5)=Prob(\hat{X}=5)$

$=Prob\{ht_b\}$

$=\frac{1}{2}\cdot\frac{1}{3}$

$=1/6$

(2) $F(x) = \text{Prob}\{\tilde{x} \leq x\}$

To draw $F(x)$ let's firs determine the intervals. $\tilde{x}$ can take the values $0, 3, 4, 5$

Then the intervals are found as

$$-\infty < x < 0$$
$$0 \leq x < 3$$
$$3 \leq x < 4$$
$$4 \leq x < 5$$
$$5 \leq x < \infty$$

Note: $F(x_i) = \sum_{x \leq x_i} p(x)$

and for the given intervals $F(x)$ is computed as

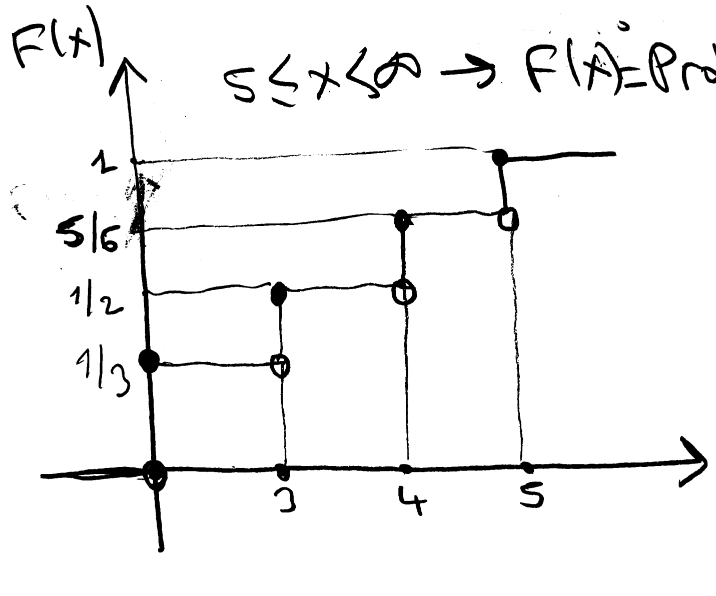
$-\infty \leq x < 0 \rightarrow F(x) = \text{Prob}\{\tilde{x} \leq x\} \rightarrow F(x) = 0$

$0 \leq x < 3 \rightarrow F(x) = \text{Prob}\{\tilde{x} \leq \underset{\text{e.g.}=1.5}{x}\} \rightarrow F(x) = p(0) = 1/3$

$3 \leq x < 4 \rightarrow F(x) = \text{Prob}\{\tilde{x} \leq \underset{\text{e.g.}=3.5}{x}\} \rightarrow F(x) = p(0) + p(3) = \frac{1}{3} + \frac{1}{6} = 3/6$

$4 \leq x < 5 \rightarrow F(x) = \text{Prob}\{\tilde{x} \leq \underset{\text{e.g.}=4.5}{x}\} \rightarrow F(x) = p(0) + p(3) + p(4) = \frac{1}{3} + \frac{1}{6} + \frac{1}{3} = 5/6$

$5 \leq x < \infty \rightarrow F(x) = \text{Prob}\{\tilde{x} \leq \underset{\text{e.g.}=6}{x}\} \rightarrow F(x) = p(0) + p(3) + p(4) + p(5) = \frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = 1$



③

Expectation, Mean, Variance, and Standard Deviation of a Random Variable

$\tilde{X} \rightarrow$ is a discrete R.V.

The expected value (also called the expectation or the mean) of a random variable $\tilde{X}$ with p.d.f $p(x)$ is computed as

$$E(\tilde{X}) = \sum_{x} x\, p(x)$$

$$m_{\tilde{X}} = E(\tilde{X})$$

used to denote the mean value of RU, $\tilde{X}$

Variance of RU. $\tilde{X}$

$$Var(\tilde{X}) = E\left[(\tilde{X} - m_{\tilde{X}})^2\right]$$

$m_{\tilde{X}} = m \rightarrow$ for simplicity

Standard deviation of $\tilde{X}$

$$Std(\tilde{X}) = \sigma_{\tilde{X}} = \sqrt{Var(\tilde{X})}$$

$$\boxed{\text{Standard Deviation} = \sqrt{\text{Variance of } \tilde{X}}}$$

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Ex: $S = \{h, t\}$

$$\tilde{X}(h) = 1$$
$$\tilde{X}(t) = -1$$

$$\text{Prob}(\tilde{X}=1) = \text{Prob}\{h\} \rightarrow p(x) \quad x=1$$
$$= \frac{1}{2}$$

$$\text{Prob}(\tilde{X}=-1) = \text{Prob}\{t\} \rightarrow p(x) \quad x=-1$$
$$= \frac{1}{2}$$

Hence $p(x) = \begin{cases} 1/2 & x=1 \\ +1/2 & x=-1 \end{cases}$

$$m_{\tilde{X}} = \sum x\, p(x)$$
$$= (1)\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right)$$
$$= 0$$

$$\text{Var}(\tilde{X}) = E\left((\tilde{X} - m_{\tilde{X}})^2\right)$$
$$= E\left((\tilde{X} - 0)^2\right)$$
$$= E\left(\tilde{X}^2\right)$$
$$= \sum_x x^2 p(x) = (-1)^2 \frac{1}{2} + (1)^2 \frac{1}{2}$$
$$= \frac{1}{2}$$

$$Std(\tilde{X}) = \sqrt{Var(\tilde{X})}$$

$$= \frac{1}{\sqrt{2}}$$

Expected Value Rule for Functions of Random Variables

$\tilde{X} \rightarrow$ R.V. $\qquad p(x) \rightarrow$ p.d.f. (prob. density function of $\tilde{X}$)

$$E(g(\tilde{X})) = \sum_x g(x)\, p(x)$$

Exe

$\tilde{X} \rightarrow$ R.V.

$$p(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in the range } [-4, 4] \\ 0 & \text{otherwise} \end{cases}$$

$E(\tilde{X}) = ?$

$E(\tilde{X}^2) = ?$ $\qquad E(\tilde{X}^3) = ?$

S/n:

$$E(\tilde{X}) = \sum_x x\, p(x)$$

$$= \sum_{x=-4}^{4} x\, p(x)$$

$$= -4.\frac{1}{8} + (-3).\left(\frac{1}{9}\right) + \dots + 4.\frac{1}{9} + 3\frac{1}{9}$$

$$= 0$$

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$$E(\tilde{X}^2) = \sum x^2 p(x)$$

$$= \sum_{x=-4}^{4} x^2 p(x)$$

$$= \frac{1}{9}\left[(-4)^2 + (-3)^2 + \cdots (3)^2 + (4)^2\right]$$

$$= \frac{60}{9}$$

$$E(\tilde{X}^3) = \sum x^3 p(x)$$

$$= \sum_{x=-4}^{4} x^3 p(x)$$

$$= \frac{1}{9}\left[(-4)^3 + (-3)^3 + (-2)^3 + \cdots (2)^3 + 3^3 + 4^3\right]$$

$$= 0/9 = 0$$

Hence $\quad E(\tilde{X}) = E(\tilde{X}^3) = 0 \qquad E(\tilde{X}^2) = \frac{60}{9}$

In general $E(\tilde{X}^k) = 0$

$k \to$ odd number

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Variances:

$$Var(\tilde{X}) = E\left((\tilde{X} - m_{\tilde{X}})^2\right)$$

$$m_{\tilde{X}} = E(\tilde{X})$$

$$Var(\tilde{X}) = E\left((\tilde{X} - m_{\tilde{X}})^2\right)$$
$$= \sum_x (x - m_{\tilde{X}})^2 \, p_{\tilde{X}}(x)$$

$$Var(\tilde{X}) = \sum_x (x - m)^2 \, p(x)$$

$$\sigma = \sqrt{Var(\tilde{X})} \rightarrow \sigma^2 = Var(\tilde{X})$$

$$Var(\tilde{X}) = E\left((\tilde{X} - m)^2\right)$$
$$= E\left(\tilde{X}^2 - 2m\tilde{X} + m^2\right)$$
$$= E(\tilde{X}^2) - 2m\underbrace{E(\tilde{X})}_{=m} + m^2$$
$$= E(\tilde{X}^2) - 2m^2 + m^2$$
$$= E(\tilde{X}^2) - m^2$$

$$Var(\tilde{X}) = E(\tilde{X}^2) - [E(\tilde{X})]^2 \qquad m = E(\tilde{X})$$

⑧ Ex:

$$p(x) = \begin{cases} 1/4 & x=-1 \\ 1/4 & x=0 \\ 1/2 & x=1 \\ 0 & \text{otherwise} \end{cases}$$

$m_{\tilde{x}} = ?$ $Var(\tilde{x}) = ?$

$\sigma_{\tilde{x}} = ?$

Sln:

$$m_{\tilde{x}} = \sum x\,p(x)$$

$$= (-1)\left(\frac{1}{4}\right) + (0)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{4}$$

$$Var(\tilde{x}) = E(\tilde{x}^2) - m^2$$

$$E(\tilde{x}^2) = \sum x^2 p(x)$$

$$= (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2}$$

$$= \frac{1}{4} + \frac{1}{2}$$

$$= \frac{3}{4}$$

$$\sigma_{\tilde{x}} = \sqrt{\frac{11}{16}}$$

$$Var(\tilde{x}) = \frac{3}{4} - \left(\frac{1}{4}\right)^2$$

$$= \frac{12}{16} - \frac{1}{16} = \frac{11}{16}$$

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Property: $\tilde{x} \to$ R.V.

$\sigma^2 = Var(\tilde{x})$

$\tilde{y} = a\tilde{x} \longrightarrow Var(\tilde{y}) = a^2 Var(\tilde{x})$

$\tilde{y} = a\tilde{x} + b \longrightarrow Var(\tilde{y}) = a^2 Var(\tilde{x})$

Proof:

$Var(\tilde{y}) = E((\tilde{y} - m_{\tilde{y}})^2)$

$m_{\tilde{y}} = E(a\tilde{x} + b)$

$= aE(\tilde{x}) + b$

$= a m_{\tilde{x}} + b$

Note: $E[g(x)] = \sum g(x) p(x)$

$\underset{g(x)}{Var(\tilde{y})} = \sum (y - m_{\tilde{y}})^2 p(x)$

$= \sum (ax + b - am_{\tilde{x}} - b)^2 p(x)$

$= \sum (ax - am_{\tilde{x}})^2 p(x) dx$

$= a^2 \sum (x - m_{\tilde{x}})^2 p(x) dx$

$= a^2 Var(\tilde{x})$

Propertys

$$\tilde{y} = a\tilde{x} + b$$

$$E(\tilde{y}) = a E(\tilde{x}) + b$$

$$\downarrow$$

$$\boxed{m_{\tilde{y}} = a\, m_{\tilde{x}} + b}$$

Some Well Known R.V.s and Their p.d.f.

The Bernoulli Random Variable

$\tilde{x} \rightarrow$ Bernoulli R.V.

$\tilde{x}$ can take two values 1 & 0

$$p(\tilde{x} = 1) = a$$

$$p(\tilde{x} = 0) = 1 - a$$

$p_{\tilde{x}}(x) \rightarrow$ p.d.f of $\tilde{x}$

The Binomial Random Variables

$\tilde{x} \rightarrow$ binomial R.V. with parameters n and p

its p.d.f is given as

$$p_{\tilde{x}}(k) = \text{Prob}(\tilde{x} = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$k = 0, 1, \ldots, n$

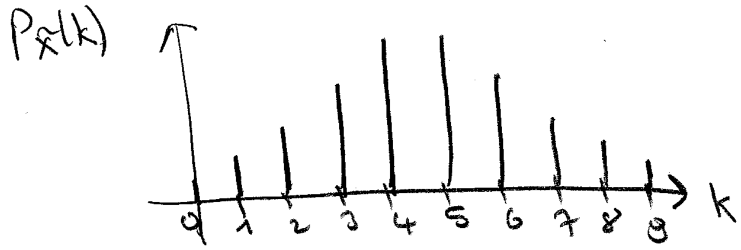
(11) Since in general $\sum p(x)=1$

then $\sum_k p(k)=1$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

for $n=9$

$1/2$

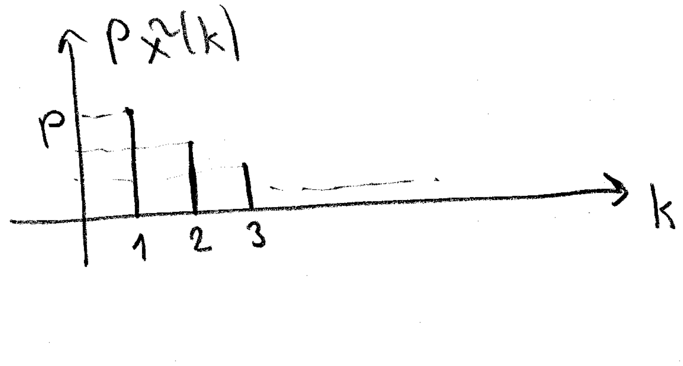


The Geometric R.V.

$\tilde{X} \rightarrow$ Geometric R.V. with p.d.f

$$P_{\tilde{X}}(k) = (1-p)^{k-1} p \quad k=1,2,\dots \quad p+q=1$$

$(1-p)^{k-1}p$ is the prob. of the sequence consisting of $k-1$ successive tails followed by a head.



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The Poisson Random Variable

$\tilde{X} \rightarrow$ Poisson R.V.

$$P_{\tilde{X}}(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad k = 0, 1, 2, \ldots$$

If $\lambda = np$, n is very large and p is very small

$$e^{-\lambda} \frac{\lambda^k}{k!} \simeq \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k} \quad k = 0, 1, \ldots n.$$

Functions of Random Variables

$\tilde{X} \rightarrow$ R.V. $\quad \tilde{Y} = g(\tilde{X})$

$P_{\tilde{X}}(x) \longrightarrow$ p.d.f. of $\tilde{X}$

$P_{\tilde{Y}}(y) \longrightarrow$ p.d.f of $\tilde{Y}$

The relation between $P_{\tilde{Y}}(y)$ & $P_{\tilde{X}}(x)$ is given as

$$P_{\tilde{Y}}(y) = \sum_{\{x \mid g(x) = y\}} P_{\tilde{X}}(x)$$

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Ex:

$\tilde{X} \to$ R.V. $\qquad \tilde{Y} = \tilde{X}^2$

$$P_{\tilde{X}}(x) = \begin{cases} 3/8 & x=-1 \\ 1/8 & x=0 \\ 4/8 & x=1 \end{cases} \qquad P_{\tilde{Y}}(y) = ?$$

Sln:

$$P_{\tilde{Y}}(y) = \sum_{\{x \mid g(x)=y\}} P_{\tilde{X}}(x)$$

$$P_{\tilde{Y}}(y) = \sum_{\{x \mid y=x^2\}} P_{\tilde{X}}(x)$$

$y = x^2 \longrightarrow$ $x=-1 \to y=1$, $x=1 \to y=1$, $x=0 \to y=0$

$P_{\tilde{Y}}(y=1) = P_{\tilde{X}}(x=-1) + P_{\tilde{X}}(x=1)$

$$P_{\tilde{Y}}(y=0) = P_{\tilde{X}}(x=0)$$

$P_{\tilde{Y}}(y=1) = \frac{3}{8} + \frac{4}{8} = 7/8 \qquad P_{\tilde{Y}}(y=0) = \frac{1}{8}$