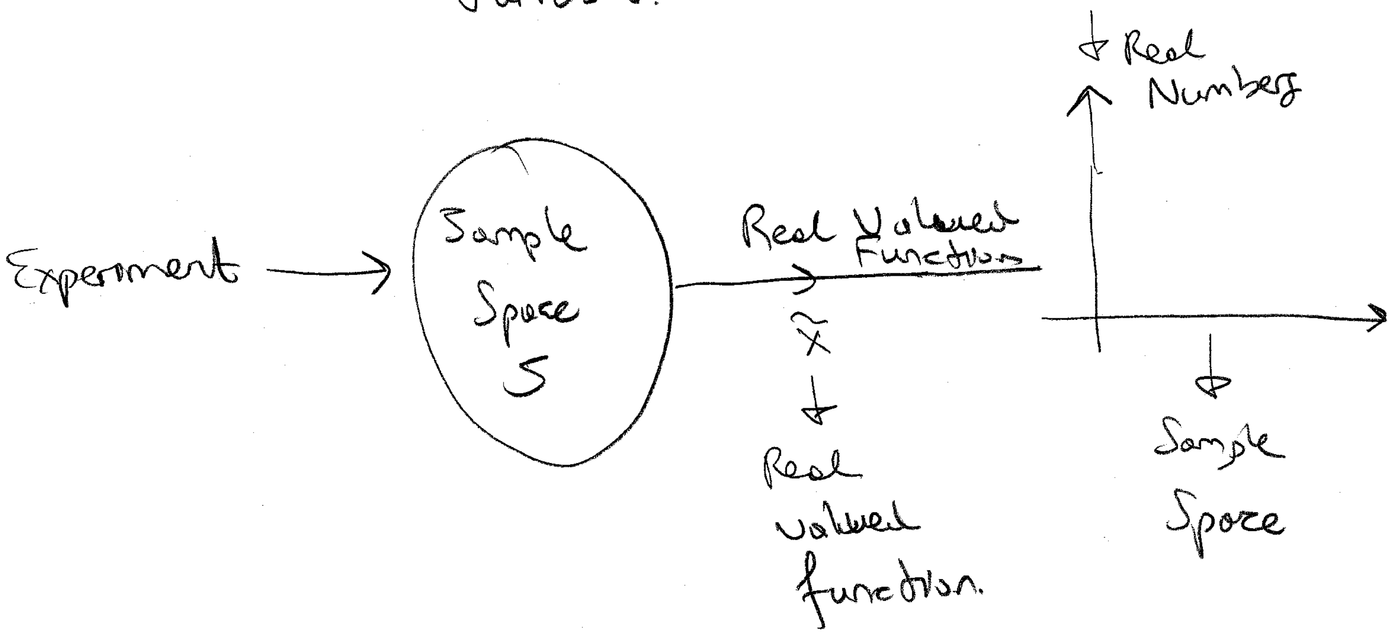
(1)

Discrete Random Variables:



A random variable is a real-valued function of the experimental outcomes.

Ex: Experiment : Coin toss

Sample Space $S = \{h, t\} = \{s_1, s_2\}$

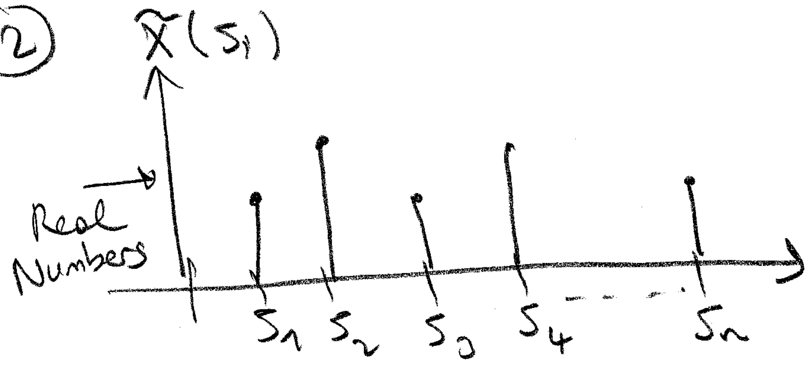
Random variable $\tilde{X}(s_i) = \begin{cases} 2 & \text{if } s_i = h \\ 3 & \text{if } s_i = t \end{cases}$

$\tilde{X}(\cdot) \rightarrow$ a random variable

i.e, it is a real valued function

In other words:

$S = \{s_1, s_2 \cdots s_n\} \rightarrow$ Sample Space $\tilde{X}(\cdot) \rightarrow$ Random variable



$\tilde{X}(s_i) \rightarrow$ a mapping from sample space to real numbers. $\tilde{X}(.)$ is named as random variable OR R.V. in short.

Ex.

A pair of coins are tossed
One fair coin, One biased coin

$S_1 = \{h, t\}$ $S_2 = \{h_b, t_b\}$

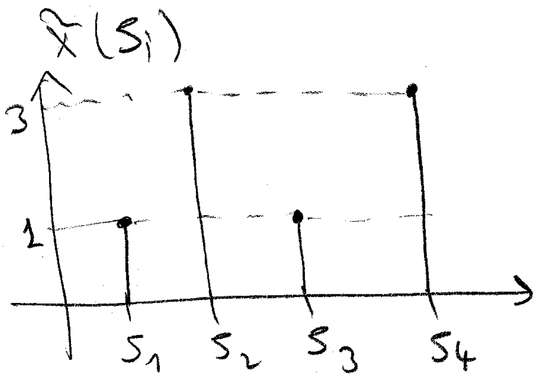
$S = S_1 \times S_2 \rightarrow S = \{h h_b, h t_b, t h_b, t t_b\}$

Simple outcomes $\leftarrow$ s_1 s_2 s_3 s_4

A real valued function defined on S as

$$\underbrace{\tilde{X}(s_i)}_{\text{simple outcome of experiment}} = \begin{cases} 1 & \text{if } s_i \text{ contains } h_b \\ 3 & \text{if } s_i \text{ contains } t_b \end{cases}$$

③



$\tilde{X}(\cdot) \rightarrow$ a random variable

i.e, a real valued function defined on sample space of an experiment.

Ex:

Experiment roll of a die

Sample space $S = \{1, 2, 3, 4, 5, 6\}$

$\tilde{X}(s_i) \longrightarrow$ (Discrete) random variable.

can be defined on S as

$$\tilde{X}(s_i) = \begin{cases} 2i-1 & \text{if } s_i \text{ is odd} \\ 2i+1 & \text{if } s_i \text{ is even} \end{cases}$$

$S = \{1, 2, 3, 4, 5, 6\}$

$s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6$

then

$\tilde{X}(s_1) = 2\times 1 - 1 \Rightarrow 2$

$\tilde{X}(s_2) = 2\times 2 + 1 \Rightarrow 5$

$\tilde{X}(s_3) = 2\times 3 - 1 \Rightarrow 5$

$\tilde{X}(s_4) = 2\times 4 + 1 \Rightarrow 9$

$\tilde{X}(s_5) = 2\times 5 - 1 \Rightarrow 9$

$\tilde{X}(s_6) = 2\times 6 + 1 \Rightarrow 13$

Main concepts related to Random Variables

- A random variable is a real-valued function of the outcome of the experiment
- A function of a random variable defines another random variable
- Each random variable has mean and variance

Notations:

$\{ s_i \mid \hat{X}(s_i) = x \}$ → is a set → Find all the simple outcomes such that they all satisfy $\hat{X}(s_i)=x$ equality

Ex: Roll of a die $S = \{1 \dots 6\}$

$$\hat{X}(s_i) = \begin{cases} 1 & \text{if } s_i \text{ is odd} \\ -1 & \text{if } s_i \text{ is even} \end{cases}$$

$A = \{ s_i \mid \hat{X}(s_i) = -1 \}$.

$\hat{X}(s_2) = -1$
$\hat{X}(s_4) = -1$
$\hat{X}(s_6) = -1$

$\Rightarrow A = \{ s_2, s_4, s_6 \}$

⑤

Ex: Two independent tosses of a fair coin.

$$S = \{hh, ht, th, tt\}$$
$$\quad\ \ s_1 \quad s_2 \quad s_3 \quad s_4$$

$$\tilde{X}(s_i) = \{\text{number of heads in } s_i\}$$

$\tilde{X} \to$ Random variable

$$\tilde{X}(s_1) = \tilde{X}(hh) = 2$$
$$\tilde{X}(s_2) = \tilde{X}(ht) = 1$$
$$\tilde{X}(s_3) = \tilde{X}(th) = 1$$
$$\tilde{X}(s_4) = \tilde{X}(tt) = 0$$

Now write the following sets explicitly

$$A = \{s_i \mid \tilde{X}(s_i) = 1\}$$

$$B = \{s_i \mid \tilde{X}(s_i) = -1\}$$

$$C = \{s_i \mid \tilde{X}(s_i) = 2\}$$

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$A = \{ s_i \mid \hat{x}(s_i) = 1 \}$ → A consists of all s_i which satisfy $\hat{x}(s_i) = 1$.

Since $\hat{x}(\overset{s_2}{ht}) = 1$

$\hat{x}(\underset{s_3}{th}) = 1$

$A = \{ ht, th \}$

In a similar manner B is found as

$B = \{ \ \}$ → empty set

C is found as

$C = \{ hh \}$

Now consider the set

$D = \{ s_i \mid \hat{x}(s_i) = 1 \text{ OR } \hat{x}(s_i) = 0 \}$

D is found as

$D = \{ ht, th, tt \}$

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Notation

$\{s_i \mid \tilde{X}(s_i)=x\} \rightarrow$ shows a set

for simplicity of notation the above expression can be written as

$$\tilde{X}=x$$

in other words

$$\underbrace{\tilde{X}=x}_{\text{a set}} \text{ means } \underbrace{\{s_i \mid \tilde{X}(s_i)=x\}}_{\text{a set}}$$

a set = a set

Notations

$\{s_i \mid \tilde{X}(s_i)\leq x\} \rightarrow$ is a set consisting of all s_i's such that they satisfy $\tilde{X}(s_i)\leq x$ inequality.

Ex: Roll of a die

$S=\{1,2,3,4,5,6\}$

$\tilde{X}(s_i)=10\times i$

$\tilde{X}(s_1)=10\times 1$
$\tilde{X}(s_2)=10\times 2$
$\vdots$
$\tilde{X}(s_6)=10\times 6$

(8) Now consider the following sets

$A = \{ s_i \mid \hat{x}(s_i) \leq 35 \}$

$B = \{ s_i \mid \hat{x}(s_i) \leq 5 \}$

$C = \{ s_i \mid 20 \leq \hat{x}(s_i) \leq 35 \}$

$D = \{ s_i \mid \hat{x}(s_i) = 40 \}$

Now lets determine the above sets

$A = \{ s_1, s_2, s_3 \} \rightarrow A = \{1, 2, 3\}$

$B = \emptyset$

$C = \{ s_2, s_3 \} \rightarrow C = \{2, 3\}$

$D = \{ s_4 \} \longrightarrow D = \{4\}$

Notations

$\{ s_i \mid \hat{x}(s_i) \leq x \} = \underbrace{\{ \hat{x} \leq x \} \Rightarrow \hat{x} \leq x}_{\text{used for simplicity.}}$

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Probability mass function for discrete random variables

$$p(x) = \text{Prob}(\overbrace{\{\tilde{x}=x\}}^{\text{A set}})$$

$$= \text{Prob}(\tilde{x}=x)$$

$p(x)$ is prob. mass function for R.V. $\tilde{x}$

$p(x)$ is also written as $p_{\tilde{x}}(x)$

<u>Ex:</u> Experiment toss of a coin two times

Sample Space $S = \{\underbrace{hh, ht, th, tt}_{\text{simple outcomes}}\}$

$\phantom{Sample Space S = \{} \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$\phantom{Sample Space S = \{} s_1 \quad s_2 \quad s_3 \quad s_4$

Random Variable

$$\tilde{x}(s_i) = \{\text{number of heads in } s_i\}$$

$\tilde{x}(hh)=2 \quad \tilde{x}(ht)=1 \quad \tilde{x}(th)=1 \quad \tilde{x}(tt)=0$

Prob. mass function of $\tilde{x}(\cdot)$

$$p(x) = \text{Prob}(\tilde{x}=x)$$

x can take values 2, 1, and 0

$$p(0) = \text{Prob}(\tilde{x}=0)$$

$$= \text{Prob}\{tt\}$$

$$= 1/4$$

(10)

$p(1) = \text{Prob}\{\hat{x} = 1\}$

$= \text{Prob}\{ht, th\}$

$= \frac{2}{4}$

$= \frac{1}{2}$

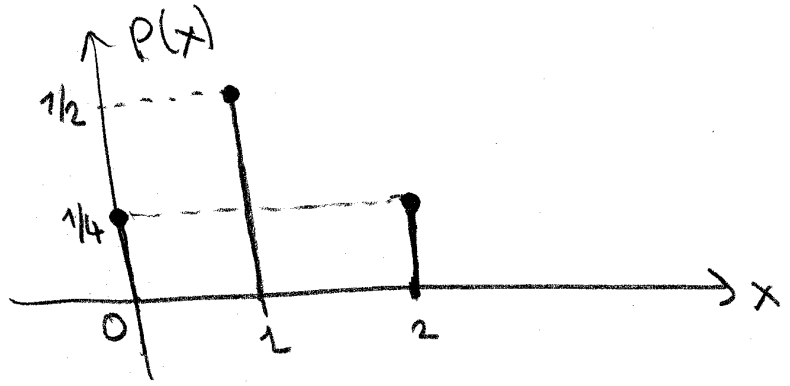
$p(2) = \text{Prob}\{\hat{x} = 2\}$

$= \text{Prob}\{hh\}$

$= \frac{1}{4}$

The graph of prob. mass function of R.V. $\hat{x}$

can be drawn as



$p(0) + p(1) + p(2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$

$= 1$

Hence if $p(x)$ is prob. mass function of R.V. $\hat{x}$ then

$$\sum_x p(x) = 1$$

For any set of real numbers R

$$p(\tilde{x} \in R) = \sum_{x \in R} p(x)$$

Ex:

For the previous example

$$p(\tilde{x}=1, \tilde{x}=0) = p(\tilde{x}=1) + p(\tilde{x}=0)$$

$$= \sum_{x \in \{0,1\}} p(x)$$

"probability mass function" of R.V $\tilde{x}$ is also called "probability density function".

i.e, prob. mass function = prob. density function

Cumulative Distribution Function of a R.V.

$\tilde{x} \to$ R.V.

Cumulative dist. funct. of R.V. $\tilde{x}$ is ~~defined~~ as

$$\underbrace{F(x)}_{\text{real number}} = \text{Prob}(\underbrace{\tilde{x} \le x}_{\text{A set}})$$

It may also written as $F_{\tilde{x}}(x) = \text{Prob}(\tilde{x} \le x)$

(12)

Ex: Experiment toss of a coin two times

Sample Space $S = \{hh, ht, th, tt\}$

Random variable

$$\tilde{x}(s_i) = \{\text{number of heads in } s_i\}$$

Question: Draw cum.. distribution function $F(x)$ of R.V. $\tilde{x}$

S/n:

$$\left.\begin{aligned}\tilde{x}(s_1) &= \tilde{x}(hh) = 2\\ \tilde{x}(s_2) &= \tilde{x}(ht) = 1\\ \tilde{x}(s_3) &= \tilde{x}(th) = 1\\ \tilde{x}(s_4) &= \tilde{x}(tt) = 0\end{aligned}\right\}$$

Consider

$\tilde{x}(s_i) = x$

OR

$\tilde{x}(s_i) \leq x$

x can take values 0, 1 and 2.

$F(x) = \text{Prob}\{\tilde{x} \leq x\}$

→ Cumulative distribution function of $\tilde{x}$

First decide the sets

$\{\tilde{x} \leq x\}$ for $x = 0$, $x = 1$, and $x = 2$

then compute their probs. and finally draw graph of $F(x)$.

(13) $\{\hat{x} \leq x\} \rightarrow$ $\{\hat{x} \leq 0\} = \{tt\}$

$\{\hat{x} \leq 1\} = \{tt, th, ht\}$

$\{\hat{x} \leq 2\} = \{tt, th, ht, hh\}$

$\{\hat{x} \leq x\} = \emptyset$ if $x < 0$

$\{\hat{x} \leq x\} = S$ if $x \geq 2$

sample space

$F(x) = \text{Prob}(\hat{x} \leq x)$

if $x < 0$ $F(x) = \text{Prob}(\hat{x} \leq x) \rightarrow \text{Prob}(\emptyset) = 0$

empty set

if $0 \leq x < 1$ $F(x) = \text{Prob}(\hat{x} \leq x)$

$= \text{Prob}(\{tt\})$

$= 1/4$

if $1 \leq x < 2$ $F(x) = \text{Prob}(\hat{x} \leq x)$

$= \text{Prob}(\{tt, th, ht\})$

$= 3/4$

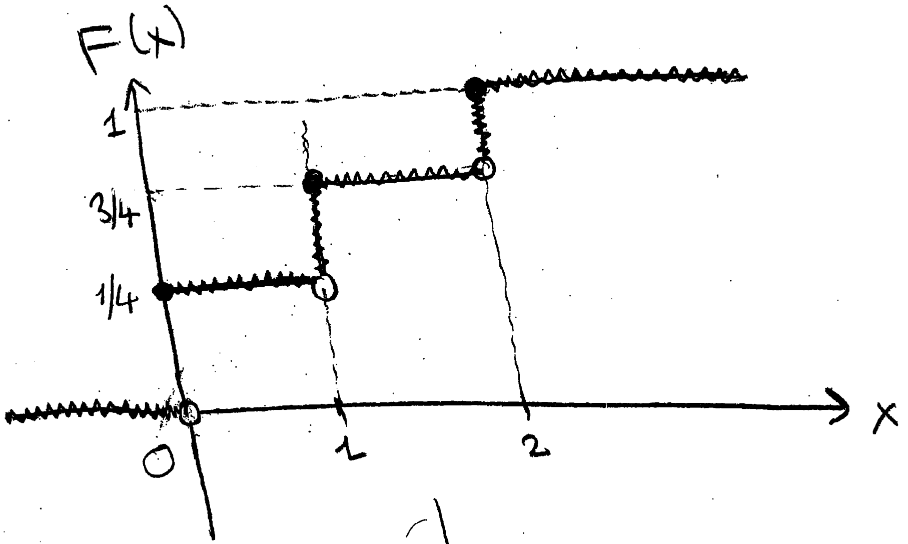
if $2 \leq x$ $F(x) = \text{Prob}(\hat{x} \leq x)$

$= \text{Prob}(S) \rightarrow$ sample space

$= 1$

(14)

Now we can draw graph of F(x)



Cumulative distribution function of R.V. X