Partitions:

Suppose we partition n distinct objects into r subsets $B_1 \dots B_r$ of size $n_1, n_2, \dots n_r$

and $n_1 + n_2 + \dots n_r = n$

The total number of choices is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\dots\binom{n-n_1\dots-n_{r-1}}{n_r}$$

$$= \frac{n!}{\cancel{(n-n_1)!}\,n_1!} \cdot \frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\,n_2!} \cdots \frac{\cancel{(n-n_1\dots-n_{r-1})!}}{(n-n_1\dots-n_{r-1}-n_r)!\,n_r!}$$

$$= \frac{n!}{n_1!\,n_2!\dots n_r!}$$

Note: If there are n different object the total number of permutations is $n!$. If n_1 objects are the same

n_2 " " " "

⋮

n_r " " " "

The total number of permutations is $\frac{n!}{n_1!\,n_2!\dots n_r!} < n!$

Ex: A six-sided die is tossed 12 times

How many distinct sequences of faces have each number appearing exactly twice. What is the prob. of obtaining such a sequence.

Sln:

$$\overbrace{11\ 22\ 33\ 44\ 55\ 66}^{12 \text{ elements}}$$

$$\frac{12!}{2!.2!.2!.2!.2!.2!} = \frac{12!}{2^6} = 7.484.400$$

There are 6^{12} possible outcomes in 12 tosses of a die

The prob of obtaining a sequence in which each face appears exactly twice is

Ex: How many different letter sequences can be obtained by rearranging letters in the word TATTOO?

Sln:

$$\frac{6!}{3!.2!.1!} = 60$$

Let $S = A_1 \cup A_2 \cup \dots \cup A_r$ → Sample Space

$A_1, A_2, A_3 \dots A_r$ disjoint sets

$A_1, A_2, \dots A_r$ is a partition of S

wit $P(A_i) = p_i$ $\quad p_1 + p_2 + \dots p_r = 1$

Repeat the experiment n times

The prob. of event {A_1 occurs k_1 times ... A_r occurs k_r times in any order}

where $k_1 + k_2 + \dots k_r = n$

$$P_n(k_1, k_2 \dots k_r) = \frac{n!}{k_1! \dots k_r!} p_1^{k_1} \dots p_r^{k_r}$$

Ex: A fair die is rolled 10 times. We shall determine the prob. that '1' shows three times and "even" shows six times.

$A_1 = \{1\}$ $\quad A_2 = \{2,4,6\}$ $\quad A_3 = \{3,5\}$ $\quad S = \{1,2,\dots 6\}$

$Prob(A_1) = p_1 = 1/6$ $\quad p_2 = 3/6$ $\quad p_3 = 2/6$

$k_1 = 3$ $\quad k_2 = 6$ $\quad k_3 = 1$

$$P_{10}(3,6,1) = \frac{10!}{3!\,6!\,1!} \left(\frac{1}{6}\right)^3 \left(\frac{1}{2}\right)^6 \left(\frac{1}{3}\right) = 0.002$$

Asymptotic Theorems:

$p(A) \rightarrow$ Prob of event $A \rightarrow p(A) = p$

Prob$\{A$ occurs k times in n trials$\}$

$$= \binom{n}{k} p^k q^{n-k}.$$

Prob$\{A$ occurs k times, $k_1 \leq k \leq k_2\}$

$$= \sum_{k=k_1}^{k_2} \binom{n}{k} p^k q^{n-k}.$$

We will develop simple approximations to evaluate these probabilities.

Gaussian Functions

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\int_{-\infty}^{\infty} g(x)\,dx = 1 \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1$$

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-(x-b)^2/2\sigma^2} dx = G\left(\frac{x_2-b}{\sigma}\right) - G\left(\frac{x_1-b}{\sigma}\right)$$

Error functions

$$\mathrm{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2} dy$$

$$= G(x) - \frac{1}{2}$$

DeMoivre Laplace Theorems

if $npq \gg 1$, then

$$\binom{n}{k} p^k q^{n-k} \cong \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$$

For k in $\sqrt{npq}$ neighborhood of np.

Stirling's formula $n! \cong n^n e^{-n} \sqrt{2\pi n}$ $n \to \infty$

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Ex: A fair coin is tossed 1000 times.

Find the prob. (P_a) that heads will show 500 times and the prob P_b that heads will show 510 times

Sln.: $k = 500 \quad \sqrt{npq} = \sqrt{250} \quad p = q = 0.5$

$$np = 500$$

$$k \in (np - \sqrt{npq},\ np + \sqrt{npq})$$

a) $$P_a = \binom{n}{k} p^k q^{n-k} \simeq \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$$

$$= \frac{1}{\sqrt{2\pi 100 \times 0.5 \times 0.5}} e^{-0}$$

$$\cong 0.0252$$

b) $k = 510 \quad P_b = \binom{n}{k} p^k q^{n-k}$

$k - np = 510 - 500 = 10$

$$\simeq \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}$$

$$= \frac{e^{-0.2}}{10\sqrt{5\pi}} = 0.0207$$

If $npq \gg 1$ $k_1 - np$ & $k_2 - np$ of the order $\sqrt{npq}$

$$\sum_{k=k_1}^{k_2} \binom{n}{k} p^k q^{n-k} \simeq \frac{1}{\sigma\sqrt{2\pi}} \sum_{k=k_1}^{k_2} e^{-(k-np)^2/2\sigma^2}$$

$$\sigma^2 = npq \gg 1$$

The Multinomial Prob. Law

$S \to$ Sample space of an experiment

$S = B_1 \cup B_2 \cup \ldots B_m$ $\quad B_1, B_2, \ldots B_m \to$ disjoint events

$$P(B_j) = p_j \qquad \mathrm{Prob}(B_j) = p_j$$

$$p_1 + p_2 + \cdots p_m = 1$$

Experiment is repeated n times.
Let k_j be the number of times event B_j occurs. Then $(k_1, k_2, \ldots k_m)$ specifies the number of each events B_j occurs. Then the prob of vector $(k_1, \ldots, k_m)$ satisfies

$$P((k_1, k_2, \ldots k_m)) = \frac{n!}{k_1! k_2! \ldots k_m!} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

Ex 3

A dart is thrown nine times at a target consisting of three areas. Each throw has a prob. of 0.2, 0.3, and 0.5 of landing in areas 1, 2, and 3, respectively. Find the prob that the dart lands exactly three times in each of the areas

Sln.: Experiment: throw dart

$S = \{1, 2, 3\}$ → sample space three areas.

$B_1 = \{1\}$ $B_2 = \{2\}$ $B_3 = \{3\}$

$S = B_1 \cup B_2 \cup B_3$ B_1, B_2, and B_3 are disjoint sets

Experiment is repeated $n = 9$ times

$$A = \begin{cases} \text{Event } B_1 \text{ occurs } k_1 = 3 \text{ times} \\ \text{Event } B_2 \text{ '' } k_2 = 3 \text{ times} \\ \text{Event } B_3 \text{ '' } k_3 = 3 \text{ ''} \end{cases} \quad \begin{aligned} P(B_1) &= 0.2 = p_1 \\ P(B_2) &= 0.3 = p_2 \\ P(B_3) &= 0.5 = p_3 \end{aligned}$$

$$P(A) = \frac{n!}{k_1!\, k_2!\, k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3}$$

$$= \frac{9!}{3!\,3!\,3!} (0.2)^3 (0.3)^3 (0.5)^3 = 0.04536$$

Ex: An urn contains 5 red, 3 green, and 4 white balls. A sample of size 8 is selected at random without replacement. The prob. that the sample space contains 2 red, 2 green 1 blue, and 3 white balls is

$$\frac{\binom{5}{2}\binom{3}{2}\binom{2}{1}\binom{4}{3}}{\binom{14}{8}}$$

Ex: A four die is rolled 10 times. Determine the prob that '2' shows 4 times and odd shows 3 times.

Notes $S=\{s_1, \dots s_m\} \to$ Sample space

$S = B_1 \cup B_2 \cup \dots B_N$ $\quad B_1, B_2, \dots,$ and B_N are disjoint sub sets

Then $\{B_1, B_2, \dots B_N\}$ form a partition of S.