Independence

A is independent of B if $P(A|B) = P(A)$

Since $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

then
$$\frac{P(A \cap B)}{P(B)} = P(A) \Rightarrow P(A \cap B) = P(A).P(B)$$

If the occurrence of two events is governed by distinct and noninteracting physical processes such events will turn out to be independent.

Ex:

Experiment: two successive rolls of a 4-sided die

16 possible outcomes each with prob. of occurrence $1/16$

a) Events: $A = \{$ 1st roll is a 1 $\}$

$B = \{$ sum of the two rolls is a 5 $\}$

Are the events A & B independent?

b) Events $A = \{$ maximum of the two rolls is 2 $\}$

$B = \{$ minimum of the two rolls is 2 $\}$

Are the events A & B independent?

Sln:

a) $A = \{(1,1),(1,2),(1,3),(1,4)\}$

$B = \{(2,3)(3,2)(1,4)(4,1)\}$

$A \cap B = \{(1,4)\}$

$P(A) = \frac{4}{16}$ $P(B) = \frac{4}{16}$ $P(A \cap B) = \frac{1}{16}$

$$P(A \cap B) \stackrel{?}{=} P(A) \cdot P(B)$$

$$\frac{1}{16} = \frac{1}{4} \times \frac{1}{4} \checkmark$$

A and B are independent events.

b) $A = \{(1,2),(2,1),(2,2)\}$ $A \cap B = \{(2,2)\}$

$B = \{(2,2),(2,3),(3,2),(2,4),(4,2)\}$

$P(A) = \frac{3}{16}$ $P(B) = \frac{5}{16}$

$P(A \cap B) = \frac{1}{16}$ $P(A \cap B) \stackrel{?}{=} P(A) \cdot P(B)$

$$\frac{1}{16} \neq \frac{3}{16} \times \frac{5}{16}$$

Hence events A & B are not independent.

Conditional Independence:

Given an event C, the events A & B are called conditionally independent if

$$P(A\cap B|C) = P(A|C)P(B|C)$$

Using definition of conditional probability and the multiplication rule

$$P(A\cap B|C) = \frac{P(A\cap B\cap C)}{P(C)}$$

$$P(A\cap B\cap C) = P(C)P(B|C)P(A|B\cap C)$$

$$P(A\cap B|C) = \frac{\cancel{P(C)}P(B|C)P(A|B\cap C)}{\cancel{P(C)}}$$

$$P(A|C)\cancel{P(B|C)} = \cancel{P(B|C)}P(A|B\cap C)$$

$$P(A|C) = P(A|B\cap C) \Rightarrow$$ A & B are conditionally independent events

This relation states that if C is known to have occured, the additional knowledge that B also occured does not change the prob. of A

Ex:

Consider two independent fair coin tosses

$H_1 = \{ 1^{st}$ toss is a head$\}$

$H_2 = \{ 2^{nd}$ toss is a head$\}$

$D = \{$ the two tosses have different results$\}$

The events H_1 & H_2 are independent. However,

$P(H_1|D) = \frac{1}{2}$ $\quad P(H_2|D) = \frac{1}{2}$

$$P(H_1 \cap H_2 | D) = 0$$

Since $P(H_1 \cap H_2|D) \neq P(H_2|D)\, P(H_2|D)$,

the events H_1 & H_2 are not conditionally independent.

Reading Assignment: Example 1.19, Page 34

Summary:

- Two events A and B are independent if
  $P(A \cap B) = P(A)\, P(B)$
  OR $P(A|B) = P(B)$
- If A & B are independent, so are A & B^c
- A & B are conditionally independent if
  $P(A \cap B|C) = P(A|C)P(B|C)$ OR $P(A|B \cap C) = P(A|C)$

- Independence does not imply conditional independence, and vice versa.

Definition of Independence of Several Events

Events $A_1, A_2, \ldots A_n$ are independent if

$$P\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i)$$ for every subset S of $\{1, 2, \ldots n\}$

Ex:

For the events A_1, A_2, and A_3 to be independent. All of the following conditions should be satisfied.

1) $P(A_1 \cap A_2) = P(A_1) P(A_2)$
2) $P(A_1 \cap A_3) = P(A_1) P(A_3)$
3) $P(A_2 \cap A_3) = P(A_2) P(A_3)$
4) $P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$

Ex:

Consider two independent rolls of a fair die

$S = \{(1,1), (1,2), (1,3), (1,5)\ (1,5), (1,6), (2,1) \ldots\} \rightarrow 36$ elements.

$A = \{$ 1st roll is 1, 2 or 3 $\}$

$B = \{$ 1st roll is 3, 4, or 5

$C = \{$ the sum of the two rolls is 9 $\}$

Compute $p(A)$, $p(B)$, $p(C)$

$p(A\cap B)$, $p(A\cap C)$, $p(B\cap C)$

$p(A\cap B\cap C)$

Decide if A, B, and C are independent events

Sln:

$A=\{(1,1)(1,2)(1,3)(1,4)(1,5)(1,6),(2,1)\ldots\} \rightarrow$ 18 elements

$B=\{(3,1)(3,2)(3,3)\ldots\ldots\} \rightarrow$ 18 elements

$C=\{(3,6)(6,3)(4,5)(5,4)\}$

$p(A)=\frac{18}{36}=\frac{1}{2}$ $\quad p(B)=\frac{18}{36}=\frac{1}{2}$ $\quad p(C)=\frac{1}{9}$

$A\cap B=\{(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)\}$

$p(A\cap B)=\frac{6}{36}=\frac{1}{6}$

$A\cap C=\{(3,6)\}$ $\quad p(A\cap C)=\frac{1}{36}$

$B\cap C=\{(3,6)(4,5),(5,4)\}$

$p(B\cap C)=\frac{3}{36}=\frac{1}{12}$

$A\cap B\cap C=\{(3,6)\}$ $\quad p(A\cap B\cap C)=\frac{1}{36}$

$$p(A\cap B\cap C)=\frac{1}{36}=\underset{1/2}{p(A)}\cdot\underset{1/2}{p(B)}\cdot\underset{1/9}{p(C)}$$

However,

$$P(A\cap B)=\frac{1}{6}\neq\frac{1}{2}\cdot\frac{1}{2}=P(A)\cdot P(B)$$

$$P(A\cap C)=\frac{1}{36}\neq\frac{1}{2}\cdot\frac{1}{8}=P(A)\,P(C)$$

$$P(B\cap C)=\frac{1}{12}\neq\frac{1}{2}\cdot\frac{1}{8}=P(B)\cdot P(C)$$

Thus, A, B, and C are not independent events.

Independent Trials and Binomial Probabilities

If an experiment involves a sequence of independent but identical stages, we say that we have a sequence of independent trials. If there are only two possible results at each stage, we say that we have a sequence of independent Bernoulli trials.

Ex: It rains or it does not rain.
coin tosses, resuls are head (H) tail (T)

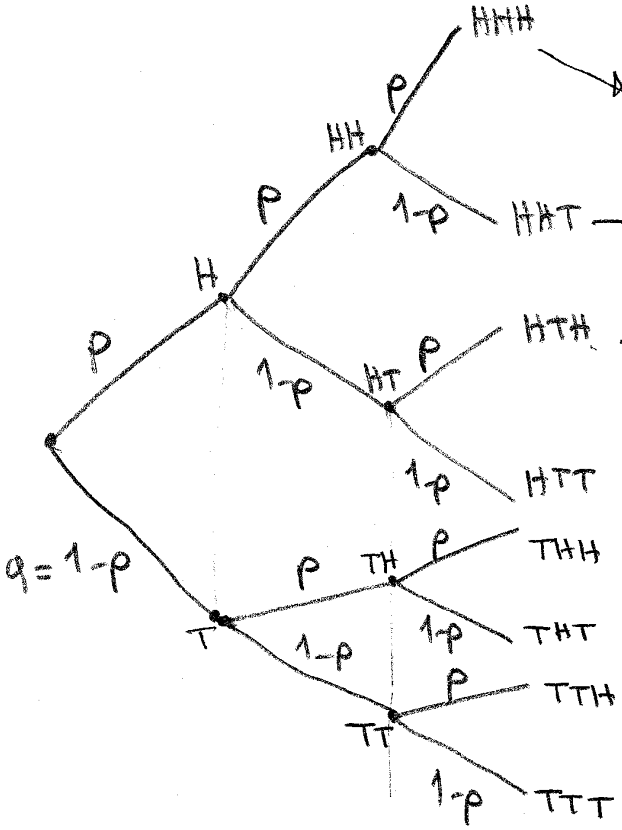
Consider the prob.

$$p(k)=\text{Prob}\left(k \text{ heads come up in an } n\text{-toss sequence}\right)$$

For instance k heads in 3 tosses

Assume biased coins $P(h)=p$ $P(t)=q$

$p+q=1$ $q=1-p$



HHH $\to$ $Prob = p\,p\,p = p^3$

HHT $\to$ $Prob = p^2(1-p)$

HTH $\to$ $Prob = p^2(1-p)$

HTT $\to$ $Prob = p(1-p)^2$

THH $\to$ $Prob = p^2(1-p)$

THT $\to$ $Prob = p(1-p)^2$

TTH $\to$ $Prob = p(1-p)^2$

TTT $\to$ $Prob = (1-p)^3$

If we sum all of the above probabilities

we get

$$p^3 + 3p^2(1-p) + 3p(1-p)^2 + (1-p)^3$$

OR $\quad p^3 + 3p^2 q + 3p q^2 + q^3 = (p+q)^3 = 1$

Now consider 2 heads in 3 tosses case

$$Prob(2 \text{ heads in 3 tosses}) = p^2(1-p) + p^2(1-p) + p^2(1-p)$$

$$= 3p^2(1-p)$$

If we generalize this

Prob(k heads in n-trials)

$$= \binom{n}{k} p^k q^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$k = 0, 1, \ldots, n$

In addition

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

Fundamental Theorems

A event $S \to$ Sample space of an experiment

Experiment is repeated n-times

$P(A) = p$

Prob(A occurs k times in n trials)

$$= \binom{n}{k} p^k (1-p)^{n-k}.$$

$$= P_n(k)$$

(10)

Ex: A fair die is rolled five times. What is the prob. that "six" will show twice.

Sln: Event $A = \{$ six will show $\}$

Experiment roll a die

$S = \{1, 2, 3, 4, 5, 6\} \rightarrow$ Sample space

$P(A) = \frac{1}{6} = p$

Experiment is repeated $n = 5$ times

What is the prob. that "six" will show $k = 2$ times

$$P(\text{"six" will show twice}) = P_n(k)$$
$$= P_5(2)$$
$$= \binom{n}{k} p^k q^{n-k}$$
$$= \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{5-2}$$
$$= \frac{5!}{3!\,2!} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$$

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Property:

Prob (the number k of occurrences of event A is between k_1 & k_2)

$$= \sum_{k=k_1}^{k_2} \binom{n}{k} p^k q^{n-k} \qquad p+q=1$$

Ex: An order of 10^4 parts is received. The prob. that a part is defective equals 0.1. What is the prob. that the total number of defective parts is a) between 100 & 1100.

b) does not exceed 1100

sln:) $A = \{$ to be defective $\} \rightarrow$ Event

$p(A) = 0.1 = p \qquad p+q=1 \rightarrow q=0.9$

$n = 10^4$ $\qquad$ $k \rightarrow$ number of defective parts

$\downarrow$ total parts

a) $p(k_1 < k < k_2) = \sum_{k=k_1}^{k_2} \binom{n}{k} p^k q^{n-k}$

$k_1 \to 100$, $k_2 \to 1100$

$$= \sum_{k=100}^{1100} \binom{10^4}{k} (0.1)^k (0.9)^{10^4-k}$$

b) $p(0 < k < 1100) = \sum_{k=0}^{1100} \binom{10^4}{k} (0.1)^k (0.9)^{10^4-k}$

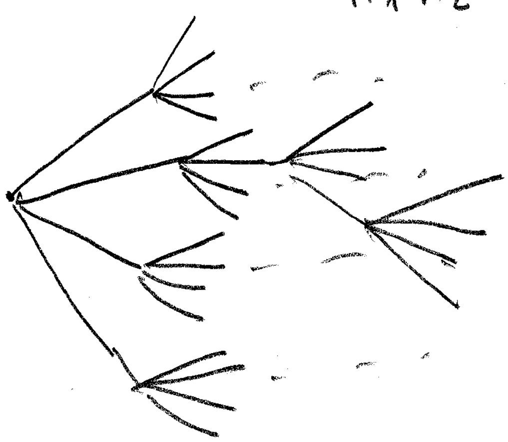
The counting Principle:

Consider a process that consists of r stages.
Suppose that

a) There are n_1 possible results for the first stage
b) For every possible result of the first stage, there are n_2 possible results at the second stage
c) In general, for all possible results of the first $i-1$ stages, there are n_i possible results at the ith stage.

Then, the total number of possible results of the r-stage process is

$$n_1 n_2 \dots n_r$$



Ex: A telephone number is a 7-digit sequence, but the first digit has to be different from 0 or 1. How many distinct telephone numbers are there?

Sln:

$$\underbrace{8}_{dig1}\ \underbrace{10}_{dig2}\cdots\underbrace{10}_{dig7} = 8\times 10^6$$

Permutations:

n-distinct objects

We wish to count the number of different ways that we can pick k out of these n objects and arrange them in a sequence

i.e; the number of distinct k-object sequences

$n \longrightarrow$ the first object can be chosen from n possible objects

$n-1 \longrightarrow$ the second object can be chosen from n-1 possible object

$\vdots$

$n-k \longrightarrow$ the k^{th} object can be chosen from n-k possible object

According to counting principle

$n(n-1)..(n-k)$ ways to from k-object sequences

$$k\text{-permutation of } n = \frac{n!}{(n-k)!}$$

$$= n(n-1)..(n-k)$$

Ex: $\{1,2,3\}$ → three object

$n=3$ $k=2$ → object sequences with two elements

12, 13, 23, 21, 31, 32 → 6 sequences

$$\frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{1.2.3}{1} = 6$$

Ex: Count the number of words that consist of four distinct letters.

Sln: English alphabet consist of 26 letters

4-permutations of 26 $= \frac{26!}{(26-4)!} = 26.25.24.23$

$= 358,800$ words.

Reading Assignment:
Example 1.27

Combinations:

There are n people and we are interested in forming a committee of k.

Ex: $n=20$ $k=3$

(a,b,c) (b,a,c) → Same Group.

Ordering is not important when forming combinations; i.e, (a, b) & (b, a) are the same.

Ex: 2-permutations of letters A, B, C, D are

AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC

the combinations of two out four of these letters are

AB, AC, AD, BC, BD, CD.

k-combinations of n objects

$$= \binom{n}{k}$$

$$= \frac{n!}{k!\,(n-k)!}$$

Ex: A batch of 50 items contains 10 defective items. Suppose that 10 items are selected at random and tested. What is the prob that exactly 5 of items tested are defective?

Sln:

$$\frac{\binom{10}{5}\binom{40}{5}}{\binom{50}{10}}$$

ways of selecting 5 items from 10 def. items

→ ways of selecting 5 items from the 40 nondefective items.

→ number of ways of selecting 10 items out of 50