(1)

$\tilde{w} = \tilde{x} + \tilde{y}$ $\quad$ $\tilde{x}, \tilde{y}, \tilde{w}$ are RVs

The Continuous Case:

CDF of $\tilde{w}$?

$$F_{\tilde{w}}(w) = P(\tilde{w} \leq w)$$

$$= P(\tilde{x} + \tilde{y} \leq w)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f_{\tilde{x}}(x) f_{\tilde{y}}(y) \, dy \, dx$$

$$= \int_{x=-\infty}^{\infty} f_{\tilde{x}}(x) \left[\int_{y=-\infty}^{w-x} f_{\tilde{y}}(y) \, dy \right] dx$$

$$= \int_{x=-\infty}^{\infty} f_{\tilde{x}}(x) F_{\tilde{y}}(w-x) \, dx$$

$$f_{\tilde{w}}(w) = \frac{d F_{\tilde{w}}(w)}{dw}$$

$$= \frac{d}{dw} \int_{x=-\infty}^{\infty} f_{\tilde{x}}(x) F_{\tilde{y}}(w-x) \, dx$$

$$= \int_{x=-\infty}^{\infty} f_{\tilde{x}}(x) \frac{d F_{\tilde{y}}}{dw}(w-x) \, dx$$

$$= \int_{x=-\infty}^{\infty} f_{\tilde{x}}(x) f_{\tilde{y}}(w-x) \, dx$$

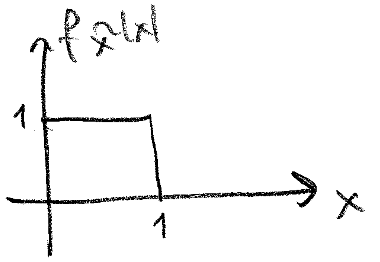
Thus, if $\tilde{w} = \tilde{x} + \tilde{y}$ then $f_{\tilde{w}}(w) = \int_{-\infty}^{\infty} f_{\tilde{x}}(x) f_{\tilde{y}}(w-x) \, dx$

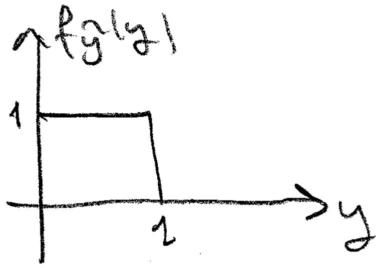
$$f_{\tilde{w}}(w) = f_{\tilde{x}}(x) \otimes f_{\tilde{y}}(y)$$ Convolution

2

Ex: $\tilde{x}$ & $\tilde{y}$ are independent and uniformly distributed in the interval $[0,1]$. Find p.d.f. of $\tilde{w} = \tilde{x} + \tilde{y}$

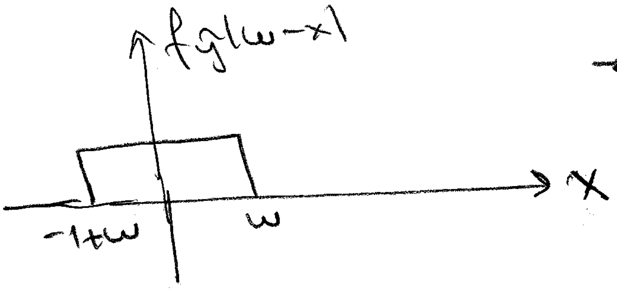
Sln:

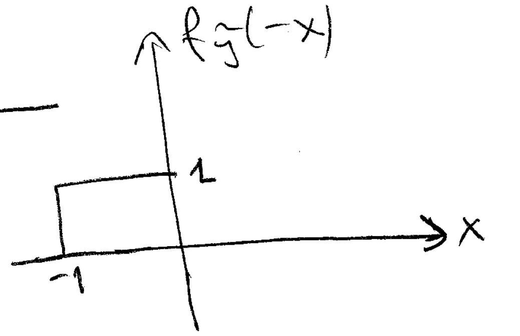




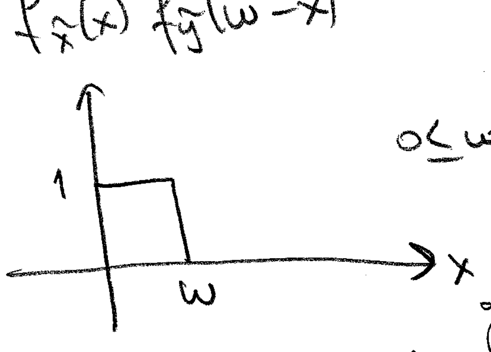
$$f_{\tilde{w}}(w) = f_{\tilde{x}}(x) \otimes f_{\tilde{y}}(y)$$

$$= \int_{-\infty}^{\infty} f_{\tilde{x}}(x)\, f_{\tilde{y}}(w-x)\, dx$$





1) $w \geq 0$

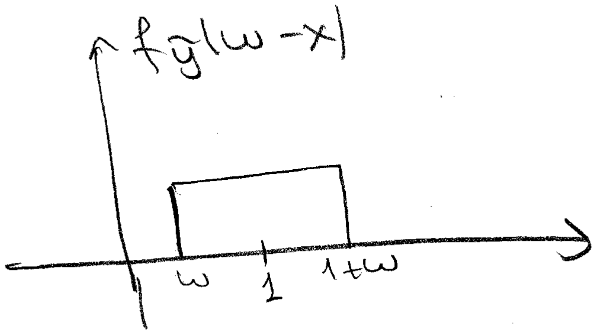


$0 \leq w < 1$

Hence, for $0 \leq w < 1$ $\quad f_{\tilde{w}}(w) = \int_{x=\infty}^{\infty} f_{\tilde{x}}(x)\, f_{\tilde{y}}(w-x)\, dx$

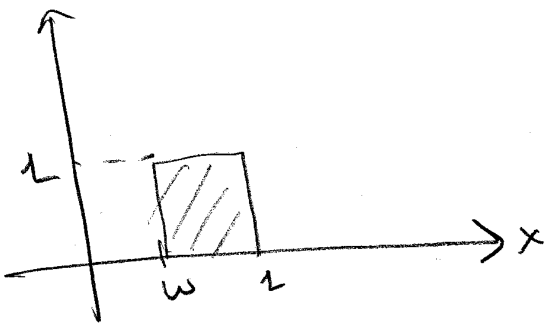
$= w$

③



$1 \leq w \leq 2$

$f_{\tilde{x}}(x)\, f_{\tilde{y}}(w-x)$

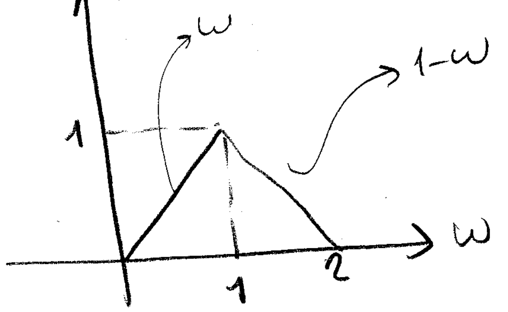


$$f_{\tilde{w}}(w) = \int_{-\infty}^{\infty} f_{\tilde{x}}(x)\, f_{\tilde{y}}(w-x)$$
$$= (1-w)\cdot 1$$
$$= 1-w \qquad 1 \leq w \leq 2$$

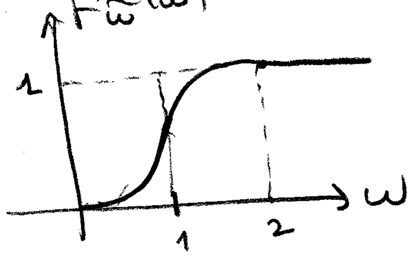
For $w<0$ and $w>2$ $\quad f_{\tilde{x}}(x)\, f_{\tilde{y}}(w-x)=0$

Hence $f_{\tilde{w}}(w) = \begin{cases} w & 0 \leq w < 1 \\ 1-w & 1 \leq w \leq 2 \end{cases}$

Graph of $f_{\tilde{w}}(w)$



C.D.F $F_{\tilde{w}}(w) = \int_{-\infty} f_{\tilde{w}}(w)\,dw \longrightarrow$



4

Conditional Expectation As a R.V.

$E[\tilde{X} | \tilde{Y} = y]$ → Conditional expectation of $\tilde{X}$ given another R.V. $\tilde{Y}$ depends on the realized value y of $\tilde{Y}$.

$$E(\tilde{X} | \tilde{Y} = y) = \sum_x x \, p_{\tilde{X}}(x|y)$$ → Discrete Case

$$E(\tilde{X} | \tilde{Y} = y) = \int_{-\infty}^{\infty} x \, f_{\tilde{X}}(x|y) \, dx$$

$$E[E(\tilde{X} | \tilde{Y})] = \sum E(\tilde{X} | \tilde{Y} = y) \, p_{\tilde{Y}}(y)$$

$$E[E(\tilde{X} | \tilde{Y})] = \int E(\tilde{X} | \tilde{Y} = y) \, f_{\tilde{Y}}(y) \, dy$$

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$$E[E(\tilde{X}|\tilde{Y})] = \underbrace{\int_{-\infty}^{\infty} E(\tilde{X}|\tilde{Y}=y)\, f_{\tilde{Y}}(y)\, dy}_{= E(\tilde{X})}$$

According to total expectation theorem

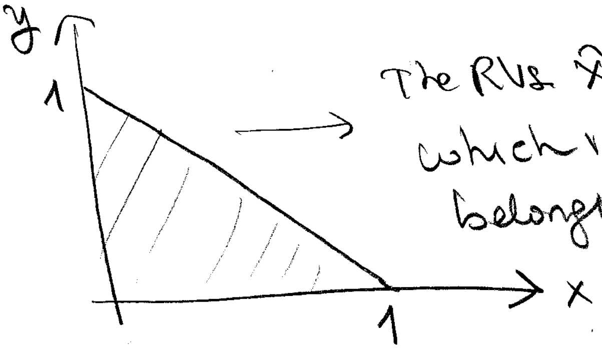
Since $E(\tilde{X}) = \sum_i E(\tilde{X}|A_i)\, P(A_i)$

$A_i = (\tilde{Y}=y)$

Law of iterated expectations:

$$E[E(\tilde{X}|\tilde{Y})] = E(\tilde{X})$$

Ex:



The RVs $\tilde{X}$ & $\tilde{Y}$ have a joint PDF which is equal to 2 for (x,y) belonging to the triangle, and zero elsewhere

Compute $E(\tilde{X}|\tilde{Y}=y)$

Find $E(\tilde{X}|\tilde{Y})$

Find $E(E(\tilde{X}|\tilde{Y}))$

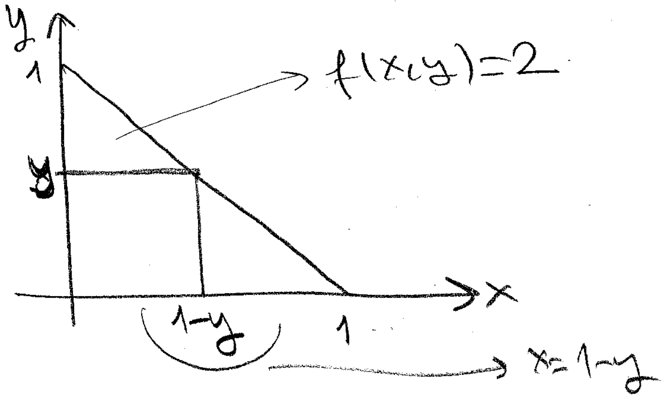
Find $E(\tilde{X})$

Sln:

$$E(\tilde{X}|\tilde{Y}=y) = \int_{-\infty}^{\infty} x\, f_{\tilde{X}|\tilde{Y}}(x|y)\, dx$$

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

⑥ $f(x|y) = \dfrac{f(x,y)}{f(y)}$



$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{1-y} 2\,dx = 2(1-y) \qquad 0 \le y \le 1$$

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{2}{2(1-y)}$$

$$= \frac{1}{1-y} \qquad 0 \le x \le 1-y$$

$$E(X|Y=y) = \int_{-\infty}^{\infty} x\, f(x|y)\,dx$$

$$= \int_0^{1-y} (x) \cdot \frac{1}{1-y}\,dx$$

$$= \frac{1-y}{2}$$

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Define $E(X|Y)$ to be the R.V. whose value is $E(X|Y=y)$ when the outcome of Y is y

Since $E(X|Y=y) = \frac{1-y}{2}$

then $E(X|Y) = \frac{1-Y}{2}$

$$E(E(X|Y)) = \int_{-\infty}^{\infty} E(X|Y=y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\frac{1-y}{2}\right) f_Y(y)\, dy$$

$$= \frac{1}{2} - \frac{1}{2}\int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \frac{1}{2}(1 - E(Y))$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \int_0^1 (y) . 2(1-y)\, dy$$

$$= 2\int_0^1 (y - y^2)\, dy$$

$$= 2\left. \frac{y^2}{2} - \frac{y^3}{3}\right|_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{3} \ //$$

$$E(E(X|Y)) = \frac{1}{3}$$

⑧

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{y=0}^{1-x} 2 .\, dy$$

$$= 2(1-x) \qquad 0 \le x \le 1$$

$$E(X) = \int x\, f_X(x)\, dx$$

$$= \int_0^1 (x)\, 2(1-x)\, dx$$

$$= (2)\ \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1$$

$$= \frac{1}{3}$$

We had found that

$$E(E(X|Y)) = \frac{1}{3}$$

$$E(X) = E(E(X|Y))$$

verified

The Conditional Variance:

$$Var(X \mid Y=y) = E\left[\left(X - E[X \mid Y=y]\right)^2 \mid Y=y\right]$$

Law of Conditional Variances:

$$Var(X) = E\left[Var(X|Y)\right] + Var\left(E(X|Y)\right)$$

Proof: $Var(X) = E\left[(X - E(X))^2\right]$

put $X - E(X) = X - E(X|Y) + E(X|Y) - E(X)$

Notes $E(g(\tilde x)\mid \tilde y = y) = \int g(x)\, f(x\mid y)\, dx$

Ex:

$$E(\tilde x^2 \mid \tilde y = y) = \int_{-\infty}^{\infty} x^2 f(x\mid y)\, dx$$

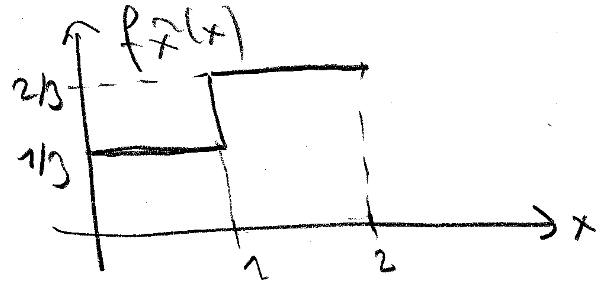
Ex:

if $E(\tilde x^2 \mid \tilde y = y) = \dfrac{y^2 - 1}{2}$

then $E(\tilde x^2 \mid \tilde y) = \dfrac{\tilde y^2 - 1}{2}$

Ex:

$$\tilde y = \begin{cases} 1 & \text{if } \tilde x < 1 \\ 2 & \text{if } \tilde x \geq 1 \end{cases}$$



$\tilde y$ is discrete R.V. $p_{\tilde y}(y) \to$ P.D.F of $\tilde y$?

Sln:

$P(\tilde y = 1) = P(\tilde x < 1)$

$P_{\tilde y}(y=1) = \int_0^1 f_{\tilde x}(x)\, dx$

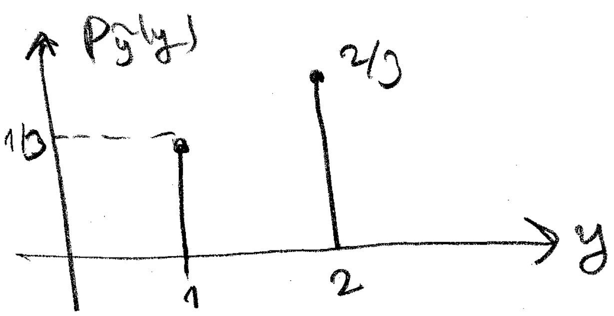
$= \frac{1}{3} \cdot 1$

$P(\tilde y = 2) = P(1 \leq \tilde x \leq 2)$

$P_{\tilde y}(y=2) = \int_1^2 f_{\tilde x}(x)\, dx$

$= \frac{2}{3}$

(10)



Example Continued

$E(\tilde{x}|\tilde{y}=y)=?$

$$E(\tilde{x}|\tilde{y}=1)=\frac{\int_0^1 x\, f(x)\, dx}{P_{\tilde{y}}(y=1)}$$

$$=\frac{\frac{1}{3}\frac{x^2}{2}\Big|_0^1}{1/3}$$

$$=1/2$$

$$E(\tilde{x}|\tilde{y}=2)=\frac{\int_1^2 x\, f(x)\, dx}{P_{\tilde{y}}(y=2)}$$

$$=3/2$$

Thus $E(\tilde{x}|\tilde{y})$ takes values $1/2$ and $3/2$ with probabilities $1/3$ and $2/3$

Mean of $E(\tilde{x}|\tilde{y})$ is $\frac{1}{2}\cdot\frac{1}{3}+\frac{3}{2}\cdot\frac{2}{3}=7/6$

$$Var(E(\tilde{x}|\tilde{y}))=E\left(\left(E(\tilde{x}|\tilde{y})-7/6\right)^2\right)$$

$$=\frac{1}{3}\left(\frac{1}{2}-\frac{7}{6}\right)^2+\frac{2}{3}\left(\frac{3}{2}-\frac{7}{6}\right)^2=2/9$$

(11)

Covariance and Correlations

The covariance of two RVs $\tilde{x}$ & $\tilde{y}$ is denoted by $cov(\tilde{x},\tilde{y})$, and is defined by

$$cov(\tilde{x},\tilde{y}) = E\left[(\tilde{x}-E(\tilde{x}))(\tilde{y}-E(\tilde{y}))\right]$$

$$cov(\tilde{x},\tilde{y}) = E\left[(\tilde{x}-m_{\tilde{x}})(\tilde{y}-m_{\tilde{y}})\right]$$

if $\tilde{x}$ & $\tilde{y}$ are independent then $E(\tilde{x}\tilde{y}) = E(\tilde{x})E(\tilde{y})$

which means that

$$Cov(\tilde{x},\tilde{y}) = E\left[\tilde{x}\tilde{y} - m_{\tilde{y}}\tilde{x} - m_{\tilde{x}}\tilde{y} + m_{\tilde{x}}m_{\tilde{y}}\right]$$

$$= \underbrace{E(\tilde{x}\tilde{y})} - m_{\tilde{y}}E(\tilde{x}) - m_{\tilde{x}}E(\tilde{y}) + m_{\tilde{x}}m_{\tilde{y}}$$

$$= \underbrace{E(\tilde{x})}_{m_{\tilde{x}}}\underbrace{E(\tilde{y})}_{m_{\tilde{y}}} - m_{\tilde{y}}m_{\tilde{x}} - m_{\tilde{x}}m_{\tilde{y}} + m_{\tilde{x}}m_{\tilde{y}}$$

$$= 0$$

Hence if $\tilde{x}$ and $\tilde{y}$ are independent RVs then $Cov(\tilde{x},\tilde{y}) = 0$

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$$Cov(X,Y) = E\left((X - m_X)(Y - m_Y)\right)$$

$$= E\left[XY - m_Y X - m_X Y + m_X m_Y\right]$$

$$= E[XY] - m_Y E(X) - m_X E(Y) + m_X m_Y$$

$$= E(XY) - m_X m_Y$$

Hence $\boxed{Cov(X,Y) = E(XY) - E(X)E(Y)}$

if $X = Y$ then

$$\underbrace{Cov(X,X)} = E(X^2) - [E(X)]^2$$

$$= Var(X)$$

Correlation Coefficient:

The correlation coefficient ρ of two RVs X and Y that have nonzero variances is defined as

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

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Moment Generating Function of a RV $\tilde{X}$

Moment generating function of RV $\tilde{X}$ is

$$M_{\tilde{X}}(s) = E\left(e^{s\tilde{X}}\right)$$

Discrete case: $M_{\tilde{X}}(s) = \sum_{x} e^{sx} p_{\tilde{X}}(x)$

Continuous case: $M_{\tilde{X}}(s) = \int_{-\infty}^{\infty} e^{sx} f_{\tilde{X}}(x)dx$

Examples

$\tilde{X}$ is exponential RV with parameter λ

$$f_{\tilde{X}}(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$M_{\tilde{X}}(s) = ?$$

Sln:

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f_{\tilde{X}}(x)\, dx$$

$$= \int_{0}^{\infty} e^{sx} \lambda e^{-\lambda x}\, dx$$

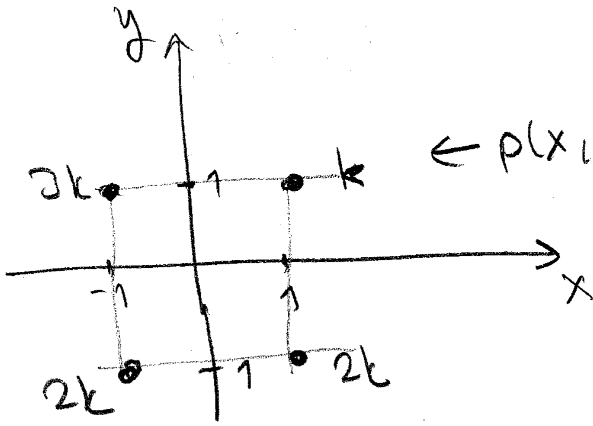
$$= \lambda \int_{0}^{\infty} e^{(s-\lambda)x}\, dx$$

$$= \lambda \left. \frac{e^{(s-\lambda)x}}{s-\lambda} \right|_{0}^{\infty} \quad (\text{if } s < \lambda)$$

$$= \frac{\lambda}{\lambda - s}$$

14

Ex 3



$\leftarrow p(x,y)$ plane for disc. RVs $\tilde{x}$ & $\tilde{y}$

a) $k=?$ b) $E(\tilde{x})$ c) $E(\tilde{y})$ e) $Var(\tilde{x})$ f) $Var(\tilde{y})$

g) $Cov(\tilde{x},\tilde{y})$, h) $E(\tilde{x}\tilde{y})=?$ i) $\rho \to$ correlation coefficient?

Sln 3

a) $\sum_{x,y} p(x,y)=1 \to 3k+k+2k+2k=1 \to 8k=1 \to k=1/8$

b) $E(\tilde{x})=\sum_{x} x\,p(x)$ $\qquad p(x)=\sum_{y} p(x,y)$

$p_{\tilde{x}}(-1)=5k \to p_{\tilde{x}}(-1)=5/8$

$p_{\tilde{x}}(1)=3k \to p_{\tilde{x}}(1)=3/8$

$E(\tilde{x})=(-1)(5/8)+(1)3/8$

$=-2/8$

c) $E(\tilde{y})=\sum_{y} y\,p_{\tilde{y}}(y)$ $\qquad p(y)=\sum_{x} p(x,y)$

$p_{\tilde{y}}(-1)=4k \to 1/2$

$p_{\tilde{y}}(1)=4k \to 1/2$

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$$E(\tilde{Y}) = (-1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right)$$

$$= 0$$

e) $$Var(\tilde{X}) = E\left((\tilde{X} - m_{\tilde{X}})^2\right)$$

$$= E(\tilde{X}^2) - m_{\tilde{X}}^2$$

$$E(\tilde{X}^2) = \sum_x x^2 p(x)$$

$$= (-1)^2 \cdot \frac{5}{8} + (1)^2 \frac{3}{8}$$

$$= 1$$

$$Var(\tilde{X}) = 1 - \left(\frac{2}{8}\right)^2$$

$$= \frac{15}{16}$$

f) $$Var(\tilde{Y}) = E\left[(\tilde{Y} - \underbrace{m_{\tilde{Y}}}_{0})^2\right]$$

$$= E\left[\tilde{Y}^2\right]$$

$$= \sum_y y^2 p_{\tilde{Y}}(y)$$

$$= (-1)^2 \cdot \frac{1}{2} + (1)^2 \frac{1}{2}$$

$$= 1 \;//$$

g) $$Cov(\tilde{X}, \tilde{Y}) = E(\tilde{X}\tilde{Y}) - \underbrace{E(\tilde{X})}_{\left(-\frac{2}{8}\right)} \underbrace{E(\tilde{Y})}_{0}$$

$$E(\tilde{X}\tilde{Y}) = \sum_x \sum_y xy \, p(x,y) = (-1)(3k) + k + 2k + (-2k)$$

$$= -2k = -1/4$$

(16) i) $\rho = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \dfrac{-1/4}{\sqrt{15/16}}$

Notes: Variance is always nonnegative

i.e $Var(X) \geq 0$

However covariance can be positive or negative

i.e, we cannot say

$Cov(X,Y) \geq 0$ wrong