Cankaya University
ECE307 - Homework-3 Solutions

Due Monday, October 31st, 2011, 15:30

1. Let $\Omega = \{Ali, Veli, 49, 50\}$. Consider a set function $\tilde{P}$, defined on subsets of Ω such that:

$$\tilde{P}(A) = \begin{cases} 1 & \text{if Ali and Veli } \in A, \\ 0.5 & \text{if either Ali or Veli } \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Does the function $\tilde{P}$ satisfy the axioms of probability?
Solution:

For a **discrete** sample space Ω, probability, $P(.)$, is a set function that assigns to every subset A of Ω a number, $P(A)$, such that:
1. $P(A) \geq 0$
2. $P(\Omega) = 1$
3. if $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$

Let's check one by one whether $\tilde{P}$ satisfies the axioms.
1. $\tilde{P}(A) \geq 0$, for all $A \subset \Omega$, by definition of $\tilde{P}(A)$. Note that it does take only 0, 0.5 or 1. Axiom 1 is satisfied.
2. $\tilde{P}(\Omega) = \tilde{P}(\{Ali, Veli, 49, 50\}) = 1$. Axiom 2 is satisfied.
3. if $A \cap B = \emptyset$ then one of the following holds:
$P(A) = 0$ and $P(B) = 0$ and $P(A \cup B) = 0$, or
$P(A) = 0$ and $P(B) = 0.5$ and $P(A \cup B) = 0.5$, or
$P(A) = 0.5$ and $P(B) = 0$ and $P(A \cup B) = 0.5$, or
$P(A) = 0.5$ and $P(B) = 0.5$ and $P(A \cup B) = 1$, or

$P(A) = 0$ and $P(B) = 1$ and $P(A \cup B) = 1$, or
$P(A) = 1$ and $P(B) = 0$ and $P(A \cup B) = 1$. Hence, Axiom 3 is also satisfied.

2. Show that $P(A^c|B) + P(A|B) = 1$.
Solution:
$P(A^c|B) + P(A|B) = \frac{P(A^c \cap B)}{P(B)} + \frac{P(A \cap B)}{P(B)} = \frac{P((A^c \cap B) \cup (A \cap B))}{P(B)}$, by Ax. 3.
$\frac{P((A^c \cup A) \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$.

3. Find $P(A|B)$ when a) $A \cap B = \emptyset$, b) $A \cap B = A$, c) $A \cap B = B$, and d) if A and B are independent events.
Solution:
a) if $A \cap B = \emptyset$ then $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0$.
b) if $A \cap B = A$ then $P(A|B) = \frac{P(A)}{P(B)}$.
c) $A \cap B = B$ then $P(A|B) = \frac{P(B)}{P(B)} = 1$.
d) if A and B are independent events, then $P(A|B) = P(A)$.
4. Consider the simplified model of communication link between Ayşe and Zehra in Fig. 1. The input from Ayşe can only take two bits, 0 or 1. Due to some sort of noise in between the two, Zehra occasionally gets the wrong bit. The probability of error is p. The probability of Ayşe sending a 0 or a 1 is 0.5.

a) Find the probability of the events:
$A = \{$Zehra receives 0$\}$, and $B = \{$Zehra receives 1$\}$.

b) Let $C = \{$error in the channel$\}$. Find $P(C|A)$ and $P(C|B)$.
Solution:

Note that:
$P(Ay = 0) = 0.5$, $P(Ay = 1) = 0.5$, $P(Z = 0|Ay = 0) = 1 - p$, $P(Z = 0|Ay = 1) = p$, $P(Z = 1|Ay = 1) = 1 - p$, $P(Z = 1|Ay = 0) = p$.
a) $A = (\{Z = 0\} \cap \{Ay = 0\}) \cup (\{Z = 0\} \cap \{Ay = 1\})$. By Ax.3, $P(A) = P(Ay = 0)P(Z = 0|Ay = 0) + P(Ay = 1)P(Z = 0|Ay = 1) = 0.5(1 - p) + 0.5p = 0.5$.
b) Similarly, $P(B) = 0.5$.
c) $P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{P(Z=0|Ay=1)P(Ay=1)}{0.5} = p$.

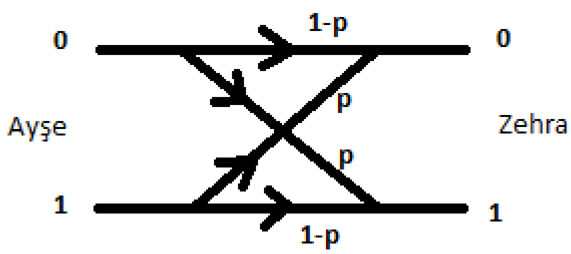

Figure 1: The noisy channel between Ayşe and Zehra. This kind of channels is known as 'Binary Symmetric Channel'.

$P(C|B) = \frac{P(C\cap B)}{P(B)} = \frac{P(Z=1|Ay=0)P(Ay=0)}{0.5} = p.$

5. A four-faced biased die is thrown with numbers from 1 to 4 on its faces. The probabilities corresponding to each of its faces are 0.2, 0.4, 0.3 and 0.1, respectively. Suppose the following four boxes are chosen at random corresponding to the numbers on the die:
Box 1 is chosen if 1 comes, which contains 5 red and 8 white marbles.
Box 2 is chosen if 2 comes, which contains 2 red and 15 white marbles.
Box 3 is chosen if 3 comes, which contains 6 red and 1 white marbles.
Box 4 is chosen if 4 comes, which contains 7 red and 11 white marbles.

a) If the marble is white, find the probability that it came from Box 3.
b) If the marble is red, find the probability that it came from Box 4.
c) If 3 comes, find the probability that a red marble is drawn.

Solution:

Note that $P(W) = P(W|1)P(1)+P(W|2)P(2)+P(W|3)P(3)+P(W|4)P(4)$.
Similarly, $P(R) = P(R|1)P(1) + P(R|2)P(2) + P(R|3)P(3) + P(R|4)P(4)$.
a)$P(3|W) = \frac{P(W|3)P(3)}{P(W)}$.
b)$P(4|R) = \frac{P(R|4)P(4)}{P(R)}$.

c)$P(R|3) = \frac{6}{7}$.

6. A pair of eight-faced dice is tossed, a coin is flipped thrice and a card is selected from a deck of 77 distinct cards. Find the number of possible outcomes.
Solution:

Total number of outcomes, $T = 8 \times 8 \times 2 \times 2 \times 2 \times 77$.